


Multi-robot Formation Control and Object Transport in Dynamic Environments via Constrained Optimization

Journal Title
XX(X):1–20
©The Author(s) 2016
Reprints and permission:
sagepub.co.uk/journalsPermissions.nav
DOI: 10.1177/ToBeAssigned
www.sagepub.com/


Javier Alonso-Mora^{1,2}, Stuart Baker¹ and Daniela Rus¹

Abstract

We present a constrained optimization method for multi-robot formation control in dynamic environments, where the robots adjust the parameters of the formation, such as size and three-dimensional orientation, to avoid collisions with static and moving obstacles, and to make progress towards their goal. We describe two variants of the algorithm, one for *local* motion planning and one for *global* path planning. The *local* planner first computes a large obstacle-free convex region in a neighborhood of the robots, embedded in position-time space. Then, the parameters of the formation are optimized therein by solving a constrained optimization, via sequential convex programming. The robots navigate towards the optimized formation with individual controllers that account for their dynamics. The idea is extended to *global* path planning by sampling convex regions in free position space and connecting them if a transition in formation is possible - computed via the constrained optimization. The path of lowest cost to the goal is then found via graph search. The method applies to ground and aerial vehicles navigating in two- and three-dimensional environments among static and dynamic obstacles, allows for reconfiguration, and is efficient and scalable with the number of robots. In particular, we consider two applications, a team of aerial vehicles navigating in formation, and a small team of mobile manipulators that collaboratively carry an object. The approach is verified in experiments with a team of three mobile manipulators and in simulations with a team of up to sixteen Micro Air Vehicles (quadrotors).

Keywords

Multi-Robot Systems, Motion Planning, Formation Control, Constrained Optimization, Sequential Convex Programming, Team of aerial vehicles, Micro Air Vehicles, Collaborative mobile manipulators, Collaborative object transport

1 Introduction

Multi-robot teams can be employed for various tasks, such as surveillance, inspection, and automated factories. In these scenarios, robots may be required to navigate in formation, for example, to maintain a communication network, to collaboratively manipulate an object or to survey an area. In this work we consider two motivating applications: formation flight for teams of unmanned aerial vehicles (UAVs) in tight spaces with static and moving obstacles, and collaborative transport of large objects by multiple mobile manipulators in automated factories and working side by side with humans and other robots.

Within the field of multi-robot navigation, formation control and reconfiguration in three-dimensional dynamic environments with moving obstacles still remains challenging. In this work we leverage efficient optimization techniques, namely quadratic programming, semi-definite programming and (non-linear) sequential quadratic programming to address this issue. Each one is employed at different stages of the proposed method for formation control among static and dynamic obstacles. These techniques provide good computational efficiency, local guarantees, and generality. Leveraging these tools, we introduce two centralized algorithms - a local motion planner and a global path planner - that enable a team of robots to navigate in formation in two-dimensional and three-dimensional environments with static and dynamic obstacles.

Given a set of target formation shapes, which serve as abstractions, our method optimizes the parameters (such as position, orientation and size) of the multi-robot formation to avoid moving obstacles and make progress towards the goal. For local motion planning, an obstacle-free region, embedded in position-time space, is first grown in a neighborhood of the robots, and then the parameters of the formation are optimized, via a constrained optimization, to remain within this region. The formation optimization method guarantees that the team of robots remains collision-free and makes progress towards the goal. To make global progress towards a goal configuration, we also present a global path planner which builds a graph of feasible formations in the environment. The graph is created by random sampling of convex regions in free space, which are kept if a valid formation exists within. A human may also provide the global path for the robots, or a desired velocity for the formation, and the robots will adapt their

¹Computer Science and Artificial Intelligence Laboratory, Massachusetts Institute of Technology, USA; ²The author is currently at the Delft Center for Systems and Control, Delft University of Technology, Netherlands

Corresponding author:

Javier Alonso-Mora, Delft University of Technology, Mekelweg 2, 2628 CD Delft, Netherlands

Email: j.alonsomora@tudelft.nl

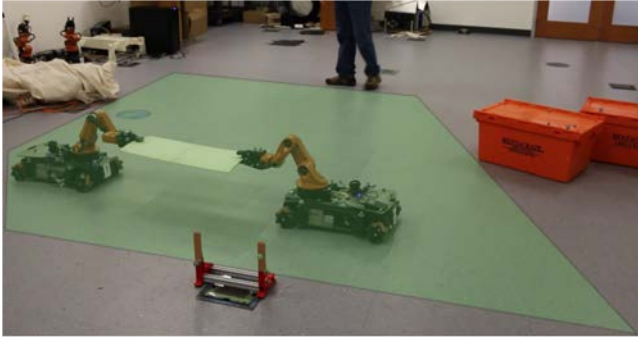


Figure 1. Two mobile manipulators collaboratively carry a rigid object. A projection of the obstacle-free convex region is superimposed in green.

configuration automatically. An example of the method for mobile manipulators is shown in Figure 1.

1.1 Contribution

The main contribution of this paper is a scalable and efficient method for navigation of a team of robots while reconfiguring their formation to avoid collisions with static and dynamic obstacles. The method applies to robots navigating in 2D and 3D workspaces and contributes:

- *Locally optimal formation control.* The parameters of the group formation are optimized online within the neighborhood of the robots via a centralized sequential convex optimization with avoidance constraints in dynamic environments.
- *A global path planner for navigating in formation.* A sampling based graph-search algorithm where convex regions in free space are sampled and connected if the intersections are traversable in formation. Sampling and non-linear optimization are combined to find a safe global path.

This work provides a working solution to a difficult problem that has not been treated at this scale before. The main strength is its ability to handle dynamic obstacles in three-dimensional environments via constrained optimization, which automatically computes the parameters of the formation to avoid collisions. Furthermore, the formation control method scales well with the number of robots, since its complexity is independent from the number of robots in the team (see Figure 2 for an example of the abstraction of the formation by its outer vertices). Finally, we validate the approach in simulations with teams of aerial vehicles and in extensive experimental demonstrations with three mobile manipulators carrying a rigid object.

In an earlier conference version of this work, (Alonso-Mora, Baker and Rus 2015), the local motion planner was introduced. In this paper we describe the approach in detail, we extend it for global path planning, and we present additional experiments with a team of three mobile manipulators collaboratively carrying an object.

The geometrical and optimization ideas of the centralized method of this paper can be combined with consensus for distributed formation control. Recently, we presented an extension (Alonso-Mora et al. 2016) to the case where the

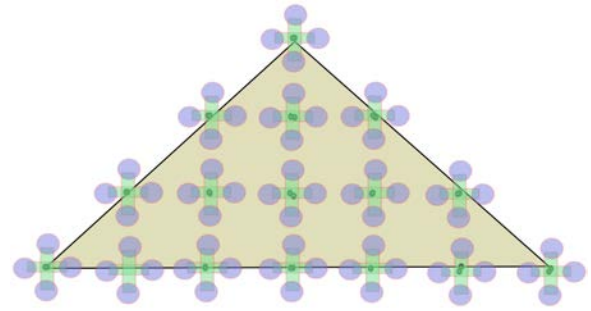


Figure 2. A triangular formation with sixteen aerial robots can be abstracted by a triangle defined by three vertices. The formation can also be defined in three-dimensional space.

robots have a reduced communication and visibility range and share information with their neighbors.

1.2 Related works

In the following we provide an overview of the related literature. In particular, we distinguish between methods for global path planning, which are typically off-line, and online methods for local motion planning and control.

1.2.1 Global path planning. Deadlock-free navigation in complex, yet *static*, environments can be achieved by computing a global path from the initial configuration to the goal configuration and a set of intermediate collision-free configurations for the team of robots. For example, Kushleyev et al. (2012) coined the problem as a mixed-integer quadratic optimization and Saha et al. (2014) relied on discretized linear temporal logic. Both methods provide global guarantees, but scale poorly with the number of robots and do not consider arbitrary formation definitions, instead they rely on squared formations.

An alternative is to randomly sample configurations in state-space to compute a set of safe configurations defining the path for the team of robots. Barfoot and Clark (2004) computed a global path for the formation via Probabilistic Roadmaps (PRM) by considering a circle enclosing the formation and a leader. Krontiris et al. (2012) later computed a PRM directly for the formation, considering its real shape and a set of templates. Our global path planner is also sampling-based, yet it differs from PRM, or pure sampling-based strategies, in that we compute both feasible formations and traversable areas in free space, which we then use to focus the sampling in unexplored regions of the workspace.

The idea of computing convex regions in free space presents similarities with early work on cell decomposition (Latombe 1991; LaValle 2006). One way to decompose the workspace into cells is to triangulate the free space. Conner et al. (2003) and Kallem et al. (2011) used such a triangulation to synthesize controllers for single robot navigation in planar environments, Ayanian et al. (2011) combined a triangulation of the environment with navigation functions to achieve multi-robot control, and Derenick and Spletzer (2007) combined a triangulation of the planar environment with second order cone programming to compute a feasible path for a circular formation. Yet, these methods are limited to planar environments.

We do not compute a typical cell decomposition of the environment, but instead rely on intersections of large convex regions to guarantee collision-free navigation in formation and reconfiguration for the team of robots. Our method builds on the work by [Deits and Tedrake \(2015\)](#), where convex polytopes were used to compute trajectories for single quadrotors. Our approach for global path planning combines sampling-based and constrained optimization techniques to explore the large configuration space. In particular, we sample overlapping convex regions in free position space and rely on a non-linear constrained optimization to compute the configuration of the robots that can occupy those spaces.

To handle moving obstacles online, we require a local motion planner which utilizes the global path for guidance and incorporates local modifications. In our local motion planner, like the global planner, we employ large convex regions, embedded in position-time space and computed in the neighborhood of the robots. The local planner computes safe motions for the team of robots in three-dimensional dynamic environments within this convex region, and is the main contribution of this work.

1.2.2 Local motion planning. A large part of formation control literature is devoted to the problem of maintaining a formation while respecting the kinematic and dynamic constraints of the robots. Examples of typical approaches for formation control include Lyapunov functions ([Ogren et al. 2001](#)), model predictive control ([Dunbar and Murray 2002](#)), flocking ([Dimarogonas and Kyriakopoulos 2005](#)) and leader-follower control ([Ren and Sorensen 2008](#)), each one with its own advantages and disadvantages. For a short review on this topic we refer the reader to ([Chen and Wang 2005](#)). In our work we do not intend to maintain a specific formation, but instead to adjust its parameters to achieve collision-free navigation in dynamic environments.

Many methods have been proposed for formation control in *obstacle-free* environments. [Balch and Arkin \(1998\)](#) employed a set of reactive behaviors. Other reactive approaches include potential fields ([Olfati-Saber and Murray \(2002\)](#), [Balch and Hybinette \(2000a\)](#) and [Sabattini et al. \(2011\)](#)) and flocking ([Tanner et al. \(2007\)](#) and [Dimarogonas and Johansson \(2008\)](#)). Two alternatives to reactive approaches are to use navigation functions ([Michael et al. \(2008\)](#)) and to synthesize controllers ([Hsieh et al. \(2008\)](#)). Other approaches exist, [Desai et al. \(2001\)](#) combined decentralized feedback laws with graph theory, [Zhou and Schwager \(2015\)](#) considered rigid formations, and [Cheah et al. \(2009\)](#) defined a formation via coverage. Shape stabilization in obstacle-free environments has also been analyzed by [Fredslund and Mataric \(2002\)](#), [Fax and Murray \(2004\)](#) and [Cortés \(2009\)](#).

Several of these approaches were also extended to *planar* environments with static obstacles. For example, social potentials were used by [Balch and Hybinette \(2000b\)](#), control of rigid body formations by [Egerstedt and Hu \(2001\)](#), abstractions to enclosing shape by [Belta and Kumar \(2004\)](#) and by [Michael and Kumar \(2008\)](#), local planning in formation space by [Kloder and Hutchinson \(2006\)](#), and controller synthesis by [Ayanian et al. \(2009\)](#). Our method is conceptually similar to ([Belta and Kumar 2004](#)) in that

we also employ an abstraction of the formation, whose dimension is independent of the number of robots. Yet, we do not synthesize controllers, but instead formulate a constrained optimization to compute the parameters of a general formation of arbitrary shape. In contrast to all of these frameworks, which were limited to planar workspaces, our method achieves collision-free motion and reconfiguration in planar and *three-dimensional* dynamic environments with moving obstacles, and therefore it applies to teams of aerial vehicles.

We formulate the problem as a constrained optimization, which can be solved online via tools for Sequential Convex Programming (SCP). Constrained optimization, and in particular semidefinite programming, was employed by [Derenick et al. \(2010\)](#) for navigating a team of robots in environments with circular obstacles, yet limited to robots moving on the plane. Sequential Convex Programming has been recently employed by [Augugliaro et al. \(2012\)](#) and [Chen et al. \(2015\)](#) to compute collision-free trajectories for multiple UAVs, although they did not consider formation control. [Morgan et al. \(2016\)](#) combined goal assignment with sequential convex programming to optimize the trajectory for a team of robots to reach a target formation, but was limited to obstacle-free environments.

For efficiency, we abstract the robot dynamics when computing the parameters of the formation, but include them in the individual robot controllers. This abstraction is similar to that of our work on pattern formation for animation display in obstacle-free environments ([Alonso-Mora et al. 2012](#)), where experiments were performed with 50 robots. Since the individual controllers do account for the robot kinematic and dynamic models, our method does apply to non-holonomic robots, as we show in simulations with teams of aerial vehicles.

1.2.3 Cooperative manipulation. Our approach for formation control applies to teams of ground and aerial robots, and it also extends to teams of cooperative manipulators collaboratively carrying an object.

One of the first approaches for collaborative object transport in *obstacle-free* environments was the use of virtual linkages by [Khatib et al. \(1996\)](#). The idea was later extended to decentralized control laws by [Sugar and Kumar \(2002\)](#) and by [Tang et al. \(2004\)](#), which enable a team of robots to accurately maintain a stable grasp in an obstacle-free environment. Static obstacles can be avoided by introducing potential functions that repel the robots from them, as shown by [Tanner et al. \(2003\)](#), but little control is then retained over the resulting configuration. Static and moving obstacles can also be avoided via constrained optimization, as shown by [Alonso-Mora, Knepper, Siegwart and Rus \(2015\)](#) for the case of deformable objects. In all of these approaches it is common to rely on force sensing to coordinate the robots.

In this work we build on these ideas and present a general, although centralized, non-convex method to compute the parameters of the formation automatically and online via sequential convex programming, which includes both global path planning and local motion planning to avoid static and dynamics obstacles. The method applies to general scenarios, specific formation types, and arbitrary number of robots. Yet, it enforces that the convex hull of the formation

remains collision-free and is therefore best suited for robots carrying convex, or near-convex, objects.

1.3 Method overview

Motion planning for a formation of robots can be seen as an instance of planning for a high-dimensional system, which can be solved with sampling-based methods. We expand on this method with a two-step approach, where the method:

- Computes convex obstacle-free regions in position space (global planner) or in position-time space (local planner), embedded in \mathbb{R}^3 or \mathbb{R}^4 .
- Executes an optimizer to compute the degrees of freedom of the formation, such as its position, size and orientation, so that the robots remain within the convex regions in free space.

This method combines some of the benefits of sampling-based methods - namely exploring a non-convex workspace and improving the quality of the solution over time - with those of local optimization methods - namely efficiently finding a local optimum in continuous space. We describe two algorithms and one extension following this idea.

In the local motion planner we rely on the notion of position-time space, where the time dimension is added to the workspace to account for moving obstacles. This is similar to the concept of configuration-time space introduced by Erdmann and Lozano-Perez (1987), but differs in that it is embedded in \mathbb{R}^4 instead of in the potentially large high-dimensional space - as would be the case for systems with many degrees of freedom. We make this natural choice explicit with the idea of planning traversable regions in position-time space and letting a non-convex optimizer compute the remaining degrees of freedom of the system to safely navigate within those traversable regions.

1.3.1 Local motion planning. Given a set of target formation shapes, our method, see Section 3, optimizes the parameters (such as position, orientation, and size) of the multi-robot formation. First, a convex obstacle-free region in position-time space is grown in a neighborhood of the robots. Second, the parameters of the formation are optimized within the convex region by solving a constrained optimization. The method guarantees that the team of robots remains collision-free and makes progress towards the goal. To make global progress towards a goal configuration, only waypoints for the formation center are required. A human may also provide the global path for the robots, or a desired velocity for the formation, and the robots will adapt their configuration automatically.

When individual robots navigate in formation, each robot independently progresses towards its assigned position in the optimal formation via a low-level planner. We employ the distributed convex optimization in velocity space by Alonso-Mora, Naegeli, Siegwart and Beardsley (2015), which avoids collisions and respects the dynamic constraints of the robot.

In Figure 3(a) we provide an overview of the method. In Figure 1 and in Figure 3(b) we show two examples of the method for mobile manipulators collaboratively carrying an object.

1.3.2 Global path planning. We also describe a method for global path planning from an initial configuration to a final configuration. The method, see Section 4, computes for the team of robots a path, and a set of safe intermediate formations. The robots avoid static obstacles and reconfigure their formation as required.

Our method presents similarities with sampling-based methods such as the Rapidly-Exploring Random Trees RRT approach by LaValle and Kuffner (2001). There, sampling was performed in configuration space and samples (i.e. configurations) and transitions were collision-checked with respect to obstacles. Here we describe an alternative, where sampling is performed in the low dimensional workspace, transitions between formations are guaranteed via convex polytopes, and safe configurations of the formation are obtained via a constrained optimization. The method introduced in this work explores the non-convex workspace and improves the quality of the solution over time thanks to sampling, while remaining efficient due to the constrained optimization and the use of convex regions, which provide a dimensionality reduction.

We create a graph of feasible formations, which connects the initial with the goal configuration. Each node in the graph is a valid configuration, which corresponds to a feasible formation embedded in free-space. Each edge between two configurations is associated with an obstacle-free convex region embedded in the workspace $\tilde{\mathcal{F}}$. The graph is created by random sampling of positions in the workspace $\tilde{\mathcal{F}}$ from which obstacle-free convex regions are grown. The parameters of valid formations within intersections of polytopes are computed via an efficient constrained optimization.

In Figure 3(c) we show an example of the method where three mobile manipulators carry an object from a start configuration (bottom left) to a goal configuration (top right). We display the first feasible path found by the algorithm, together with the sampled convex regions and the intermediate formations within the intersections.

1.3.3 Generality. We will first describe, in Sections 3 and 4, the method for a team of mobile robots navigating in a formation that can change shape via isomorphic transformations. We will then, in Section 5, describe an alternative formation definition for mobile manipulators collaboratively carrying objects. The method is general and can be adapted to other high-dimensional problems or formation definitions. The core idea is to generate convex, obstacle-free regions and then optimize the parameters of the formation (i.e. the degrees of freedom of the high-dimensional configuration) such that the robots are fully contained in the convex region. The only requirements to adapt the method are (a) a function $\mathcal{V}(\mathbf{z})$ that converts configurations \mathbf{z} to the outer vertices \mathbf{v} of the formation, and (b) a way to compute derivatives of the position of those outer vertices with respect to the configuration \mathbf{z} (unless they are computed numerically).

In this paper we describe two applications:

- **Formation control:** The configuration of the robot team is given by the 3D position, size and 3D orientation of the formation, i.e. $\mathbf{z} \in \mathbb{R}^3 \times \mathbb{R}_+ \times SO(3)$. Given a template formation, such as a square, the outer vertices

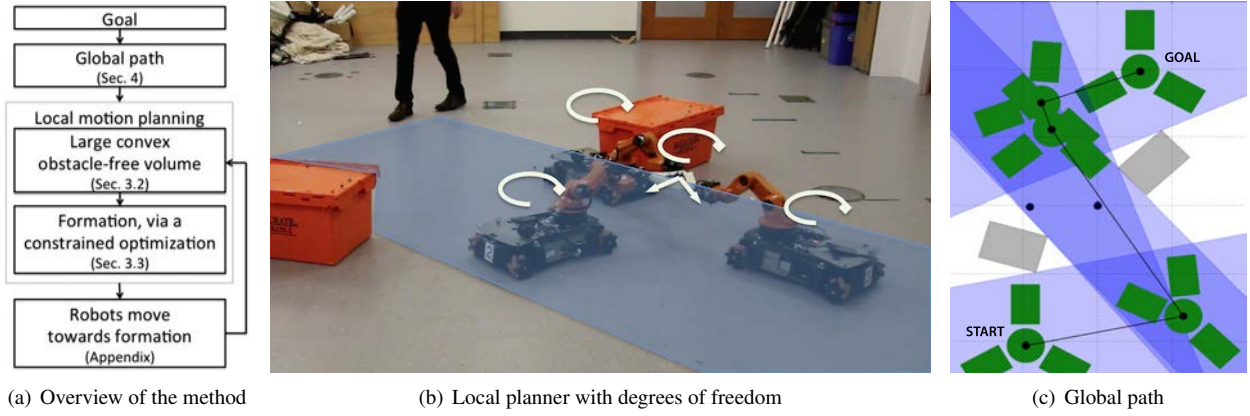


Figure 3. (a) Schematic overview of the method. Given a goal location for the team of robots, we first compute a global path from the start to the goal location, see Section 4. Then, the robots navigate along this path with continuous replanning via a local motion planner, which is described in Section 3. (b) Example with three mobile manipulators collaboratively carrying an object, see Section 5 for the extension of the method to cooperative object transport. In this case, the robots can rotate around their grasping point and a projection of the obstacle-free convex region is shown in blue. (c) Global path to navigate from the formation on the bottom left to the formation on the top right, see Section 4. Obstacles are shown in gray and the robots' formation in green. Obstacle-free convex regions (blue) connect the start with the goal configuration. Two regions are grown from the start and goal positions and the two intermediate regions are grown from random samples in the workspace (black dots). An optimized formation, in green, was computed for each of the two intersections between adjacent regions. The resulting path (solid black line) connects the start and the goal configurations and traverses through the convex regions.

are computed via an isomorphic transformation. This is our running example.

- Collaborative transportation with mobile manipulators: The configuration of the robot team is given by the 2D position and orientation of the object that the robots carry, the orientation of the n robots around their grasping points and the length of their arms, i.e. $\mathbf{z} \in \mathbb{R}^2 \times SO(2)^{n+1} \times \mathbb{R}^n$. The outer vertices of the robots and object can be computed with their shapes and a set of rigid body transformations defined by the configuration. This is described as an extension of the method in Section 5.

An advantage of the method is that planning is decoupled into: (a) finding convex regions in the lower-dimensional free position-time space (\mathbb{R}^4) and (b) efficiently optimizing the configuration of the team of robots within those convex regions. This comes at the expense of completeness, since in our approach we require that the robot team maintains a formation that does not intersect with obstacles, i.e. the robots can not maintain a formation while letting an obstacle pass through. In the event of dynamic obstacles, the team may break the formation to let a moving obstacle pass through, and come back to the original formation as soon as there is enough free room.

1.3.4 Organization. In Section 2 we introduce the notation and we describe the formation definition and the method to compute convex regions. The algorithm for local motion planning is detailed in Section 3, followed by the global path planner in Section 4. In Section 5 we introduce an extension of the method for transportation of an object by multiple manipulators. Section 6 presents experimental results with mobile manipulators and simulations with aerial vehicles. Finally, Section 7 provides a discussion of the method and Section 8 concludes this paper.

2 Preliminaries

In Table 1 we provide a list of the main symbols and variables employed in this paper.

2.1 Robots

Consider a team of robots navigating in formation. For each robot $i \in \mathcal{I} = \{1, \dots, n\} \subset \mathbb{N}$, its position at time t is denoted by $\mathbf{p}_i(t) \in \mathbb{R}^3$. In the next two sections, we consider all robots to have the same dynamic model and cylindrical non-rotating shape of radius r and height $2h$ in the vertical dimension. Denote the volume occupied by a robot at position \mathbf{p} by $\mathcal{A}(\mathbf{p}) \subset \mathbb{R}^3$.

For an alternative description of the robots, where cylindrical shape is not required, refer to the extension for rectangular mobile manipulators in Sec. 5.

2.2 Obstacles

Consider a set of static obstacles $\mathcal{O} \subset \mathbb{R}^3$ defining the global map. Denote by $\bar{\mathcal{O}}$ the set \mathcal{O} dilated by half of the robot's volume, formally

$$\bar{\mathcal{O}} = \{\mathbf{p} \in \mathbb{R}^3 \mid \mathcal{A}(\mathbf{p}) \cap \mathcal{O} \neq \emptyset\} \quad (1)$$

Moving obstacles within the field of view of the robots can be accounted for. Denote by $\mathcal{I}_D = \{1, \dots, n_d\} \subset \mathbb{N}$ the list of moving obstacles. For moving obstacle $j \in \mathcal{I}_D$ and time t , we denote by $\mathcal{D}_j(t) \subset \mathbb{R}^3$ the volume that it occupies, and

$$\bar{\mathcal{D}}_j(t) = \{\mathbf{p} \in \mathbb{R}^3 \mid \mathcal{A}(\mathbf{p}) \cap \mathcal{D}_j(t) \neq \emptyset\}, \quad (2)$$

its dilation by half of robot's volume. For predicted future positions we employ the constant velocity assumption.

2.3 Obstacle-free workspace

The obstacle-free workspace, accounting only for static obstacles, is denoted by

$$\bar{\mathcal{F}} = \mathbb{R}^3 \setminus \bar{\mathcal{O}} \subset \mathbb{R}^3. \quad (3)$$

Symbol	Definition	First appears
n / m	Number of robots / template formations	Sec. 2
\mathbf{p}	Position or point in the workspace (typically in \mathbb{R}^3)	Sec. 2
$\mathcal{A}(\mathbf{p})$	Volume occupied by a robot	Sec. 2
τ	Time horizon of the local motion planner	Sec. 2
$\bar{\mathcal{O}} / \bar{\mathcal{O}}$	Static obstacles / Dilated	Sec. 2
$\mathcal{D}_j / \bar{\mathcal{D}}_j$	Dynamic obstacle / Dilated	Sec. 2
$\mathcal{O}^\tau(t_o) / \bar{\mathcal{O}}^\tau(t_o)$	Union of static and dynamic obstacles for time $[t_o, t_o + \tau]$ / Dilated	Eq. (4)
$\bar{\mathcal{F}} / \bar{\mathcal{F}}$	Obstacle-free static workspace (workspace minus static obstacles / Dilated)	Eq. (3)
$\bar{\mathcal{F}}^\tau(t_o) / \bar{\mathcal{F}}^\tau(t_o)$	Obstacle-free workspace in position-time / Dilated	Eq. (5)
\mathcal{P}	Obstacle-free convex polytope (several variants)	Sec. 2.4, 2.3
$\mathbf{r}_j^f / \mathbf{r}_{0,j}^f$	Position of robot j in the optimized/template formation f	Sec. 2.5
$\mathbf{v}_j^f / \mathbf{w}_j^f$	Outer vertex of the optimized/template formation f	Sec. 2.5
$\mathbf{t} / \mathbf{s} / \mathbf{q}$	Position (translation), size and orientation (quaternion) of a formation	Sec. 2.5
\mathbf{z}	Configuration, i.e. optimization parameters of a formation	Sec. 2.5
$\mathcal{V}(\mathbf{z}, f)$	Set of outer vertices of formation f with configuration \mathbf{z}	Eq. (8)
\mathbf{g}	Goal position for the centroid of the robot team	Sec. 3
$\bar{\mathbf{s}} / \bar{\mathbf{q}}$	Desired size/orientation for the formation	Sec. 3
$G = \{V, E\}$	Graph containing convex regions and target formations	Sec. 4

Table 1. Table of main symbols used in the method.

For current time t_o , and time horizon τ of the motion planner, denote the union of static and dynamic obstacles seen by the robots by

$$\bar{\mathcal{O}}^\tau(t_o) = \bar{\mathcal{O}} \times [0, \tau] \cup \bigcup_{\substack{t \in [0, \tau] \\ j \in \mathcal{I}_D}} \bar{\mathcal{D}}_j(t_o + t) \times t \subset \mathbb{R}^4, \quad (4)$$

where \times denotes the Cartesian product of two sets, and with a slight abuse of notation we denote by variable $t \in \mathbb{R}$ the set $\{t\}$ containing a single point.

The position-time obstacle-free workspace is then

$$\bar{\mathcal{F}}^\tau(t_o) = \mathbb{R}^3 \times [0, \tau] \setminus \bar{\mathcal{O}}^\tau(t_o) \subset \mathbb{R}^4. \quad (5)$$

2.4 Directed obstacle-free convex region

The first building block of the proposed algorithm is to, given an obstacle map and a starting point, compute an obstacle-free convex polytope. We employ a fast iterative method, by [Deits and Tedrake \(2014\)](#), to compute large convex polytopes in free position space, i.e. $\mathcal{P}(\bar{\mathcal{F}}) \subset \bar{\mathcal{F}}$, or in free position-time space, i.e. $\mathcal{P}(\bar{\mathcal{F}}^\tau(t_o)) \subset \bar{\mathcal{F}}^\tau(t_o)$. With an abuse of notation, we may refer to this polytope by \mathcal{P} , and recall that for local motion planning (Section 3) we embed it in position-time space and for global path planning (Section 4) we embed it in position space. The method consists of two recurrent steps: (a) it computes the separating hyperplanes between an ellipsoid E and the obstacles $\bar{\mathcal{O}}$ via a quadratic optimization; and (b) it computes, via a semi-definite program, the largest ellipsoid E contained within the convex polytope \mathcal{P} . This polytope \mathcal{P} can be described by the union of hyperplanes, see Fig. 4 for an example. We extend the method by

- considering a set of points In , potentially $\{\mathbf{p}_1, \dots, \mathbf{p}_n\}$, to be contained within \mathcal{P} . The iterative algorithm breaks at convergence or when $In \not\subset \mathcal{P}$,
- growing the region \mathcal{P} towards a desired point \mathbf{g}_{dir} . This is achieved by initializing E to be the minimal ellipsoid such that $\{In, \mathbf{g}_{dir}\} \subset E$. The point \mathbf{g}_{dir} is

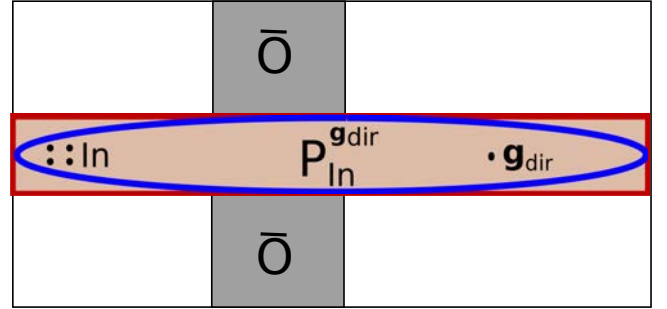


Figure 4. Example of a convex directed polytope $\mathcal{P}_{In}^{\mathbf{g}_{dir}}(\bar{\mathcal{F}})$ (in red) and its associated ellipsoid (blue) in an environment with two static obstacles (gray). The polytope contains both the In points and the target point \mathbf{g}_{dir} .

typically set to the goal position for the robot team $\mathbf{g}(t_f)$, and must also be contained within \mathcal{P} . If no solution exists, we evaluate alternative points via a linear search between \mathbf{g}_{dir} and the centroid of the points in In .

If the polytope is embedded in position space, we denote by $\mathcal{P}_{In}^{\mathbf{g}_{dir}}(\bar{\mathcal{F}})$ the resulting convex polytope, which contains the points in In and does not intersect any of the obstacles, i.e. satisfies $In \subset \mathcal{P}_{In}^{\mathbf{g}_{dir}}(\bar{\mathcal{F}}) \subset \bar{\mathcal{F}}$ and which is grown in the direction of \mathbf{g}_{dir} as described in the previous paragraph.

Definition 1. Directed polytope. We refer to a polytope $\mathcal{P}_{In}^{\mathbf{g}_{dir}}(\bar{\mathcal{F}}) \subset \mathbb{R}^3$, embedded in $\bar{\mathcal{F}}$ as a directed polytope, towards \mathbf{g}_{dir} . Analogously for a polytope $\mathcal{P}_{In}^{\mathbf{g}_{dir}}(\bar{\mathcal{F}}^\tau(t_o)) \subset \mathbb{R}^4$, embedded in $\bar{\mathcal{F}}^\tau(t_o)$.

2.5 Definition of the formation

For an alternative description of the formation, refer to the extension for mobile manipulators in Sec. 5.

We consider a pre-defined set of $m \in \mathbb{N}$ template formations, such as square, line, or T. Each template formation $f \in \mathcal{I}_f = \{1, \dots, m\}$ is given by a set of robot positions $\{\mathbf{r}_{0,1}^f, \dots, \mathbf{r}_{0,n}^f\}$ and a set of outer vertices

$\{\mathbf{w}_1^f, \dots, \mathbf{w}_{n_f}^f\}$ relative to the center of rotation (typically the centroid) of the formation, where n_f denotes the number of outer vertices defining formation f . The set of vertices represents the convex hull of the robot's positions in the formation, thus reducing the complexity for formations with a large number of robots. See Fig. 5(a) for an example.

Further denote by d_f the minimum distance between any given pair of robots in the template formation f . Template formations can be defined by a human designer or automatically computed for optimal representation of a target shape as showed by [Alonso-Mora et al. \(2012\)](#).

A formation is then defined by an isomorphic transformation, which includes an expansion $s \in \mathbb{R}_+$, a translation $\mathbf{t} \in \mathbb{R}^3$ and a rotation $R(\mathbf{q})$ described by a unit quaternion $\mathbf{q} \in SO(3)$, its conjugate denoted by $\bar{\mathbf{q}}$. With this formation definition, the configuration for the team of robots is fully defined by $\mathbf{z} = [\mathbf{t}, s, \mathbf{q}] \in \mathbb{R}^3 \times \mathbb{R}_+ \times SO(3)$.

Given the configuration \mathbf{z} , and template formation ID f , the robot positions and outer vertices of the resulting formation are computed by

$$\begin{aligned} \mathbf{r}_i^f &= \mathbf{t} + s R(\mathbf{q}) \mathbf{r}_{0,i}^f, \quad \forall i \in [1, n], \\ \mathbf{v}_j^f &= \mathbf{t} + s R(\mathbf{q}) \mathbf{w}_j^f, \quad \forall j \in [1, n_f], \end{aligned} \quad (6)$$

where the rotation in $SO(3)$ is given by the quaternion operation

$$\begin{bmatrix} 0, & R(\mathbf{q}) \mathbf{w}_j^f \end{bmatrix}^T = \mathbf{q} \times \begin{bmatrix} 0, & \mathbf{w}_j^f \end{bmatrix}^T \times \bar{\mathbf{q}}. \quad (7)$$

For template formation f and configuration \mathbf{z} we denote the set of outer vertices by

$$\mathcal{V}(\mathbf{z}, f) = [\mathbf{v}_1^f, \dots, \mathbf{v}_{n_f}^f]. \quad (8)$$

In the exposition of the method we rely on this definition for the formation, but the method is general and can be applied to alternative definitions, as shown in Sec. 5 for the case of several manipulators transporting a rigid object.

3 Local Motion Planning

The local motion planner computes the optimal parameters, i.e. the configuration, of the formation, in a neighborhood of the robots, via a constrained non-linear optimization. For a given template formation $f \leq m$, the vector of optimization

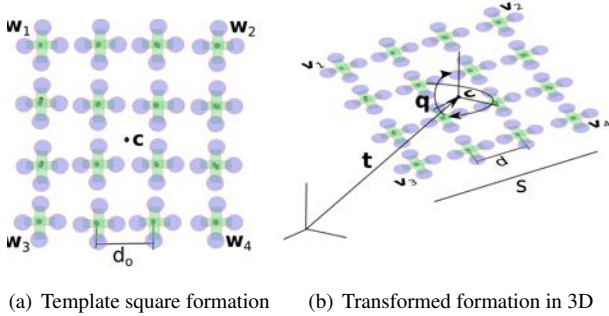


Figure 5. (a) Example of a template square formation with sixteen UAV robots with the four vertexes defining the convex hull. (b) The formation can be transformed with a translation \mathbf{t} , a 3D rotation \mathbf{q} and a size s isomorphic transformation.

variables, i.e. the configuration, is denoted by $\mathbf{z} = [\mathbf{t}, s, \mathbf{q}] \in \mathbb{R}^3 \times \mathbb{R}_+ \times SO(3)$.

Denote by $\mathbf{g}(t) \in \mathbb{R}^3$ the goal position for the centroid of the formation at time t . This goal position, and a target orientation $\bar{\mathbf{q}}$ and size \bar{s} , can be given by a human operator or a global planner, as described in the forthcoming Section 4.

For an alternative formulation of the optimization, see the extension for mobile manipulators in Section 5.

3.1 Algorithm overview

To make progress towards the goal position while avoiding obstacles, the local planner computes a target formation and the required motion of the robots for a given time horizon $\tau > 0$, which must be longer than the required time to stop. Denote the current time by t_o and $t_1 = t_o + \tau$.

Our method consists of the following steps.

- Compute a large convex polytope \mathcal{P} contained in free position-time space, such that the robots are inside it, i.e. $\mathbf{p}_i(t_o) \in \mathcal{P} \subset \bar{\mathcal{F}}^\tau(t_o), \forall i \in \mathcal{I}$, and that is directed towards the goal $\mathbf{g}(t_1)$. This is described in Sec. 3.2.
- Compute the optimal formation f^* and configuration \mathbf{z}^* such that the outer vertices $\mathcal{V}(\mathbf{z}^*, f^*)$ are contained within \mathcal{P} and the distance between the formation's centroid and the goal $\mathbf{g}(t_1)$ is minimized. The parameters of the formation are optimized subject to a set of constraints via a centralized sequential convex optimization described in Sec 3.3. In this computation the robot's dynamics are abstracted.
- In a faster loop, described in Sec. 3.5, the robots are optimally assigned to target positions in the formation and move towards them employing a low level local planner that generates collision-free inputs that respect the robot's dynamics. In particular, we build on a distributed convex optimization described by [Alonso-Mora, Naegeli, Siegwart and Beardsley \(2015\)](#), extended to account for static obstacles in a seamless way.
- If no feasible formation exists in a neighborhood of the robots, we search for the parameters of a target formation near the goal position. In this case, the robot team splits and each robot navigates independently towards its assigned position in the target formation.

3.2 Obstacle-free convex region

First, the obstacle-free space in position-time $\mathbb{R}^3 \times [0, \tau] \subset \mathbb{R}^4$ is obtained, accounting for static and dynamic obstacles.

For the given time horizon τ consider the union of static and dynamic obstacles $\bar{\mathcal{O}}^\tau(t_o)$ and the associated position-time obstacle-free workspace $\bar{\mathcal{F}}^\tau(t_o)$, as described in Eq. (5).

Following Sec 2.4, we compute two convex polytopes:

- $\mathcal{P}_{f_o \rightarrow g}$ in free position-time space, which contains all the robots at their current positions and initial time, i.e. $[\mathbf{p}_1(t_o), \dots, \mathbf{p}_n(t_o)] \times 0$, and which is directed towards the formation's goal at the time horizon, i.e. $[\mathbf{g}(t_1) \times \tau] \in \mathbb{R}^4$. Formally,

$$\mathcal{P}_{f_o \rightarrow g} := \mathcal{P}_{[\mathbf{p}_1(t_o), \dots, \mathbf{p}_n(t_o)] \times 0}^{[\mathbf{g}(t_1) \times \tau]}(\bar{\mathcal{F}}^\tau(t_o)). \quad (9)$$

Algorithm 1 Local motion planning

Given: Union of static and dynamic obstacles in position-time space $\mathcal{O}^\tau(t_o) \subset \mathbb{R}^4$ at the initial time. The goal position $\mathbf{g}(t_1) \in \mathbb{R}^3$ for the centroid of the formation at time t_1 and desired size \bar{s} and orientation $\bar{\mathbf{q}}$.

Compute: Target configuration \mathbf{z}^* and formation f^* . Collision-free motion for the team of robots for up to the time horizon τ .

———— Main process ————

- 1: **while not** converged **do**
- 2: Compute large convex polytope $\mathcal{P} \subset \bar{\mathcal{F}}^\tau(t_o)$ in a neighborhood of the robots.
- 3: Compute optimal configuration \mathbf{z}^* and formation f^* , such that the outer vertices $\mathcal{V}(\mathbf{z}^*, f^*)$ are contained within \mathcal{P} .
- 4: **end while**
- Second parallel process, at high frequency ————
- 5: **while not** converged **do**
- 6: Assign robots to target positions in the formation.
- 7: Navigate towards the target formation.
- 8: **end while**

- $\mathcal{P}_{o \rightarrow g}$ in free position-time space, which contains the centroid of the robots' current positions and initial time, i.e. $[\sum_{i \in \mathcal{I}} \mathbf{p}_i(t_o)/n] \times 0 \in \mathbb{R}^4$, and which is directed towards the goal. Formally,

$$\mathcal{P}_{o \rightarrow g} := \mathcal{P}_{[\sum_{i \in \mathcal{I}} \mathbf{p}_i(t_o)/n] \times 0}^{[\mathbf{g}(t_1) \times \tau]}(\bar{\mathcal{F}}^\tau(t_o)). \quad (10)$$

This polytope may not contain all of the robots at their current positions.

A representative example of these regions (projected in \mathbb{R}^2) is shown in Figure 6. In general, we consider the convex polytope

$$\mathcal{P} = \mathcal{P}_{f_o \rightarrow g} \cap \mathcal{P}_{o \rightarrow g}, \quad (11)$$

which:

- Guarantees that the transition to the new formation will be collision-free, since $\mathcal{P} \subset \mathcal{P}_{f_o \rightarrow g}$ and all the robots are within the convex region $\mathcal{P}_{f_o \rightarrow g}$.
- Is likely to make progress in future iterations, since $\mathcal{P} \subset \mathcal{P}_{o \rightarrow g}$, which is directed towards the goal.

Once the robots are within this intersection they can make progress towards the goal within $\mathcal{P}_{o \rightarrow g}$ in a collision-free manner. If $\mathcal{P} = \emptyset$, an alternative convex region is selected as described in the forthcoming Sec. 3.4.

We rely on a representation of the collision-free convex polytope \mathcal{P} given by its equivalent set of linear constraints

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^4 \mid A\mathbf{x} \leq \mathbf{b}, \text{ for } A \in \mathbb{R}^{n_l \times 4}, \mathbf{b} \in \mathbb{R}^{n_l}\}, \quad (12)$$

where n_l denotes the number of faces of \mathcal{P} .

3.3 Non-linear optimization

We formulate a constrained non-linear optimization to compute a locally optimal formation f^* and the configuration \mathbf{z}^* for the team of robots.

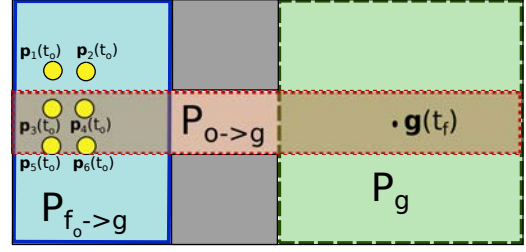


Figure 6. Example of convex regions computed with the method described in Sec 2.4 for a scenario with six yellow disk robots and two (grey) squared obstacles. The convex regions are: $\mathcal{P}_{f_o \rightarrow g}$ in blue solid border, $\mathcal{P}_{o \rightarrow g}$ in red dotted border and \mathcal{P}_g (defined in Sec. 3.4) in green dashed border.

3.3.1 Optimization cost We minimize a weighted sum of the deviation with respect to the formation's goal $\mathbf{g}(t_1)$, a desired size \bar{s} and a desired rotation $\bar{\mathbf{q}}$. The cost term is then

$$J_f(\mathbf{z}) = w_t \|\mathbf{t} - \mathbf{g}(t_1)\|^2 + w_s \|s - \bar{s}\|^2 + w_q \|\mathbf{q} - \bar{\mathbf{q}}\|^2 + c_f, \quad (13)$$

where w_t, w_s, w_q are design weights, and c_f is the predefined cost for formation type $f \in \mathcal{I}_f$.

3.3.2 Constraints Constraints are introduced to guarantee that all the robots in the formation are within the convex polytope (C_1) and to satisfy a minimum inter-robot distance (C_2) in order to avoid collisions between the robots in the team. Recalling Sec. 2.5 the constraints are then given by

$$\begin{aligned} C_1 &\equiv \mathcal{V}(\mathbf{z}, f) \times t_1 \subset \mathcal{P} \equiv \bigcup_{j=1}^{n_f} \{A[\mathbf{t} + s R(\mathbf{q}) \mathbf{w}_j^f] \times t_1\}^T \leq \mathbf{b} \} \\ C_2 &\equiv \{s \geq 2 \frac{\max(r, h)}{d_f}\} \end{aligned} \quad (14)$$

where C_1 contains a constraint for each vertex \mathbf{w}_j^f of the convex hull of the template formation f and implies that the robots do not collide with any obstacle. The constraint C_2 guarantees inter-robot collision avoidance since d_f is the minimum inter-robot distance for the template formation, recall Sec. 2.5, and the transformation applied to the formation is isomorphic. *

For planar formations, the additional constraints $\mathbf{q} \cdot [0, 1, 0, 0]^T = 0$ and $\mathbf{q} \cdot [0, 0, 1, 0]^T = 0$ may also be imposed to ensure rotation only occurs around the vertical axis.

3.3.3 Non-linear program For a template formation $f \in \mathcal{I}_f$ the optimal configuration \mathbf{z}_f^* is found by solving the non-linear optimization

$$\begin{aligned} \mathbf{z}_f^* &= \arg \min_{\mathbf{z}} J_f(\mathbf{z}) \\ \text{s.t. } &\mathcal{V}(\mathbf{z}, f) \times t_1 \subset \mathcal{P} \quad (C_1) \\ &\{s \geq 2 \frac{\max(r, h)}{d_f}\} \quad (C_2) \end{aligned} \quad (15)$$

We employ the non-linear solver SNOPT by Gill et al. (2002), which internally executes a sparse Sequential Quadratic Program and converges to feasible local minimum of the cost function.

*In our implementation we represent quaternions by vectors in \mathbb{R}^4 of unit norm and consider the additional constraint $C_3 = \{\|\mathbf{q}\|^2 = 1\}$.

The derivatives of the cost function (Eq. 13) and constraints (Eq. 14) are given by

$$\partial J_f(\mathbf{z})/\partial \mathbf{z} \equiv 2[w_t(\mathbf{t} - \mathbf{g}(t_1)), w_s(s - \bar{s}), w_q(\mathbf{q} - \bar{\mathbf{q}})]. \quad (16)$$

$$\begin{aligned} \partial C_1/\partial \mathbf{z} &\equiv \bigcup_{j=1}^{n_f} [A, A R(\mathbf{q}) \mathbf{w}_j^f, s A \partial R(\mathbf{q}) \mathbf{w}_j^f / \partial \mathbf{q}] \\ \partial C_2/\partial s &\equiv d_f \end{aligned} \quad (17)$$

where $\partial R(\mathbf{q}) \mathbf{w}_j^f / \partial \mathbf{q}$ is the Jacobian of Eq. (7)[†].

We set the initial point for the optimizer to

$$\mathbf{z}_{ini} = [\mathbf{g}(t_1), 2 \max(r, h)/d_f, \mathbf{q}_{ini}], \quad (18)$$

where the initial quaternion is chosen to be equal to the quaternion addition of the desired orientation and a small random quaternion, i.e. $\mathbf{q}_{ini} = \bar{\mathbf{q}} + \mathbf{q}_{rand}$. The additional term is included to avoid singularities of the optimizer when some components of $\bar{\mathbf{q}}$ are zero.

If the constrained optimization of Eq. (15) is solved for each template formation, the index f^* of the locally optimal formation is then given by

$$f^* = \arg \min_f J_f(\mathbf{z}_f^*). \quad (19)$$

This formation definition and its associated non-linear optimization are given as an example, but the framework is general and can be applied to other problem instances. In Sec 5 we describe how to apply this method for manipulation of rigid objects.

3.4 Iterations if problem is infeasible

The method, as described in the previous subsections, results in a formation and its configuration for the robots' team. It may occur though, that the robots make no progress towards the goal (deadlock) or that the optimization is infeasible, for example if the region \mathcal{P} defined in Eq. (11) is too small and no feasible formation fits inside. In that case, one may search for a feasible formation using an alternative region. For a representative example see Fig. 6.

First, we would repeat the optimization using the convex region $\mathcal{P}_{f_o \rightarrow g}$. If a valid target formation is found in this step, or in the original optimization with polytope \mathcal{P} , the transition is guaranteed to be collision-free, thanks to the convexity of the polytope $\mathcal{P}_{f_o \rightarrow g}$ which contains the current position of all the robots.

If this additional step is also unfeasible, then no formation may exist such that the robots can continue navigating in formation towards the goal. In that case the optimization can be repeated using the polytope $\mathcal{P}_{o \rightarrow g}$ or directly the polytope $\mathcal{P}_g := \mathcal{P}_{[\mathbf{g}(t_1), \tau]}(\bar{\mathcal{F}})$ that contains the formation's goal. Note though that these two polytopes do not contain the current robot positions. If a formation is found, the robots move individually, i.e. separately, towards their respective positions in the target formation. In this case, the formation is likely broken during the transition, but, this gives further flexibility to the method and the robots to navigate in formation whenever possible, via splitting and merging.

3.5 Individual planning towards target formation

The result of the computation of Sec. 3.3 is a target formation f^* and configuration \mathbf{z}^* . The associated set of target robot positions \mathbf{r}_j , for all $j \in \mathcal{I}$ can be computed with Eq. (6).

In this section, we describe the local planner that links the centralized formation optimization with the physical robots. At each step of the execution, at higher frequency than that of computing a new formation, the following steps are executed.

3.5.1 Goal assignment Robots are optimally assigned to the target positions \mathbf{r}_j with the objective of minimizing the sum of squared traveled distances. The optimal assignment $\sigma : \mathcal{I} \rightarrow \mathcal{I}$ is

$$\min_{\sigma} \sum_{i \in \mathcal{I}} \|\mathbf{p}_i - \mathbf{r}_{\sigma(i)}\|^2 \quad (20)$$

Following Alonso-Mora et al. (2012), this assignment can be centrally computed with the optimal Hungarian algorithm by Kuhn (1955), used in this work, or a suboptimal auction algorithm by Bertsekas (1988), which scales well with the number of robots.

3.5.2 Collision avoidance To control the individual robots in the team and to avoid collisions between them, we employ the collision avoidance algorithm introduced by Alonso-Mora, Naegeli, Siegwart and Beardsley (2015). We employ the same constraints to avoid moving obstacles and to respect the kinematic model of the robots. To better handle environments with static obstacles, we include additional linear constraints defined by a convex polytope in free space, computed in a neighborhood of the robot. For details refer to the Appendix. This method, a convex optimization in velocity space, is well suited for our application. The formation control algorithms described in this paper are agnostic to the low level controller and a different one could be employed.

4 Global Path Planning

Given an initial and a target configuration for the robot team, the global path planner computes a feasible path and intermediate formations to connect them. This is achieved by combining a sampling-based approach with constrained non-linear optimization, the idea being to sample in a low dimensional space (workspace) and letting the optimizer compute the remaining degrees of freedom.

In particular, we create a graph where each node is a feasible formation and which contains the initial and the goal configuration. An edge between two nodes, or formations, is a convex region in free space, which contains both formations. An edge provides the means to transition between two nodes in the graph. An example was shown in Figure 3(c).

The approach can be applied to a single user-defined formation (i.e. square) or when multiple formations are given. In the later, reconfiguration between formation's shape would be allowed. In an abuse of notation, throughout this section we drop the subindex $f \in \mathcal{I}_f$, consider a single formation f and refer to a polytope $\mathcal{P}_p(\bar{\mathcal{F}})$ embedded in the free position space, i.e. $\mathcal{P}(\bar{\mathcal{F}}) \subset \bar{\mathcal{F}}$, by \mathcal{P} . This is in contrast to the local planning approach, where we embedded the

[†]We employ the implementation within the Drake toolbox, available in RobotLocomotion/drake/

convex polytope in the free position-time space. Therefore, we do not take into account moving obstacles in the global path planning.

4.1 Algorithm

Consider the obstacle-free workspace $\bar{\mathcal{F}}$ defined by Eq. (3), the start position \mathbf{s} of the formations' centroid and its goal position $\mathbf{g} \in \mathbb{R}^3$. In Algorithm 2 we describe the proposed anytime method to compute a path for the robot team to navigate in formation from \mathbf{s} to \mathbf{g} .

A graph $G = \{V, E\}$ is incrementally created. Each vertex in the vertexes list V is given by the configuration \mathbf{z} of a feasible formation. Each edge in the edge list E connects two nodes, i.e. valid configurations $\mathbf{z}_1, \mathbf{z}_2$ for the team of robots, if a convex region \mathcal{P} exists such that the robots in both configurations are fully contained in the convex polytope.

We keep a list of existing polytopes P . And, for each polytope $\mathcal{P} \in P$ we keep a list $L_{\mathcal{P}}$ of configurations for which the team of robots is fully contained within the polytope. The node list is initialized with the initial \mathbf{z}_s and final \mathbf{z}_g configurations with centroids \mathbf{s} and \mathbf{g} . Analogously, the polytopes list is initialized with the convex regions \mathcal{P}_s and \mathcal{P}_g , which contain the initial and final configurations respectively.

The method proceeds by drawing random samples in the workspace (\mathbb{R}^3 for aerial vehicles). Each random sample $\mathbf{p} \in \mathbb{R}^3$ is rejected if it is inside an obstacle or one of the polytopes in the list P . If $\mathbf{p} \in \bar{\mathcal{F}} \setminus \bigcup_{\mathcal{P} \in P} \mathcal{P}$ then,

- A large convex polytope $\mathcal{P}_{\mathbf{p}}(\bar{\mathcal{F}}) \subset \bar{\mathcal{F}}$ is grown from \mathbf{p} following the method of Section 2.4.
- For each polytope $\mathcal{P} \in P$ that intersects $\mathcal{P}_{\mathbf{p}}$, we compute a configuration \mathbf{z} and formation f for the team of robots such that the formation's vertices are fully contained within the intersection of both polytopes $\mathcal{V}(\mathbf{z}, f) \subset \mathcal{P}_{\mathbf{p}} \cap \mathcal{P}$ and that minimizes the squared distance from its centroid to \mathbf{g} . The configuration \mathbf{z} and formation f are computed via a non-linear optimization analogous to that of Section 3.3. For polytope \mathcal{P} we denote this function by $\text{formation}(\mathcal{P})$. If a valid configuration exists, it is added to the node list.
- If a valid configuration \mathbf{z} is added to the node list, then (a) an edge $\{\mathbf{z}, \mathbf{z}_i, \mathcal{P}_{\mathbf{p}}\}$ is added for all configurations $\mathbf{z}_i \in L_{\mathcal{P}_{\mathbf{p}}}$ and (b) an edge $\{\mathbf{z}, \mathbf{z}_i, \mathcal{P}\}$ is also added for all configurations $\mathbf{z}_i \in L_{\mathcal{P}}$. Recalling the previous section, it is guaranteed that the team of robots can navigate between both formations through the associated convex polytope.

A feasible solution is found as soon as a path (or sequence of connected vertexes in the graph G) is found which connects the initial position with the goal position of the formation's centroid. If we let the algorithm run longer, for example until most of the free space is covered by convex regions, the best path so far is found via graph search. This is, by computing the path of lowest cost in the graph G . For each edge E between two configurations \mathbf{z}_1 and \mathbf{z}_2 we define its cost by the distance between the centroids of \mathbf{z}_1 and \mathbf{z}_2 .

Algorithm 2 Global path planning

- 1: Given: obstacle field \mathcal{O} , start configuration \mathbf{z}_s with centroid \mathbf{s} and destination \mathbf{g} for formation's centroid.
 - 2: Returns: a path \mathcal{T} of feasible configurations (formations) from \mathbf{s} to \mathbf{g} .
We describe a bidirectional graph search. The method can be adapted to a tree search.
 - 3: Initialize empty graph $G = \{V, E\}$: $V = \emptyset$; $E = \emptyset$
 - 4: Initialize empty polytope list $P = \emptyset$
 - 5: Add the initial configuration to the node list $V \leftarrow \mathbf{z}_s$
 - 6: Generate $\mathcal{P}_s, \mathcal{P}_g \subset \bar{\mathcal{F}}$ from \mathbf{s} and \mathbf{g}
Add them to the polytope list
 - 7: $P \leftarrow \mathcal{P}_s, P \leftarrow \mathcal{P}_g$
Compute a valid configuration in the goal region
 - 8: $\mathbf{z}_g = \text{formation}(\mathcal{P}_g)$
 - 9: Add the goal configuration to the node list $V \leftarrow \mathbf{z}_g$
Create lists of valid configurations for both polytopes
 - 10: $L_{\mathcal{P}_s} = \{\mathbf{z}_s\}, L_{\mathcal{P}_g} = \{\mathbf{z}_g\}$
Check if the start and goal can be connected
 - 11: **if** $\exists \mathbf{z} = \text{formation}(\mathcal{P}_s \cap \mathcal{P}_g)$ **then**
 - 12: $V \leftarrow \{\mathbf{z}\}$
 - 13: $E \leftarrow \{\mathbf{z}_s, \mathbf{z}, \mathcal{P}_s\}$
 - 14: $E \leftarrow \{\mathbf{z}_g, \mathbf{z}, \mathcal{P}_g\}$
 - 15: **end if**
The following search loop can be executed until the first feasible path is found, until the whole space $\bar{\mathcal{F}}$ is explored or up to a maximum time bound.
 - 16: **while** not end **do**
 - 17: Generate random sample $\mathbf{p} \in \bar{\mathcal{F}} \setminus (\bigcup_{i \in P} \mathcal{P}_i)$
 - 18: Generate polytope $\mathcal{P}_{\mathbf{p}} \subset \bar{\mathcal{F}}$ grown from \mathbf{p}
Try to create new node
 - 19: **if** $\exists \mathbf{z} = \text{formation}(\mathcal{P}_{\mathbf{p}})$ **then**
 - 20: $L_{\mathcal{P}_{\mathbf{p}}} = \emptyset$
Try to create new nodes and edges
 - 21: **for** $\mathcal{P} \in P$ **do**
 - 22: **if** $\exists \mathbf{z}_1 = \text{formation}(\mathcal{P} \cap \mathcal{P}_{\mathbf{p}})$ **then**
 - 23: **for** $\mathbf{z}_i \in L_{\mathcal{P}}$ **do**
 - 24: $E \leftarrow \{\mathbf{z}_1, \mathbf{z}_i, \mathcal{P}\}$
 - 25: **end for**
 - 26: **for** $\mathbf{z}_i \in L_{\mathcal{P}_{\mathbf{p}}}$ **do**
 - 27: $E \leftarrow \{\mathbf{z}_1, \mathbf{z}_i, \mathcal{P}_{\mathbf{p}}\}$
 - 28: **end for**
 - 29: $V \leftarrow \mathbf{z}_1$
 - 30: $L_{\mathcal{P}_{\mathbf{p}}} \leftarrow \{\mathbf{z}_1\}; L_{\mathcal{P}} \leftarrow \{\mathbf{z}_1\}$
 - 31: **end if**
 - 32: **end for**
 - 33: $P \leftarrow \mathcal{P}_{\mathbf{p}}$
 - 34: **end if**
 - 35: **return** $\mathcal{T} = \text{shortestPath}(G)$
 - 36: **end while**
-

4.2 Execution in composition with the local motion planner

To navigate the team of robots from the initial to the goal configuration a global path consisting of a sequence $\mathcal{T} = \{\mathbf{z}_s, \dots, \mathbf{z}_g\}$ of configurations is first obtained via the global path planning algorithm of the previous section. Each configuration $\mathbf{z}_w \in \mathcal{T}$ in the sequence, provides an

Algorithm 3 Function: $\mathbf{z} = \text{formation}(\mathcal{P})$ Input: Convex polytope $\mathcal{P} \subset \bar{\mathcal{F}}$.Output: A valid configuration \mathbf{z} such that $\mathcal{V}(\mathbf{z}, f) \subset \mathcal{P}$, or \emptyset .

- 1: **if** $\mathcal{P} = \emptyset$ **then**
- 2: **return** \emptyset
- 3: **else**
- 4: **return** result of a non-linear optimization analogous to that of Section 3.3 with convex polytope \mathcal{P} and minimizing $J(\mathbf{z})$ the deviation to a target configuration at \mathbf{g} , i.e.

$$\begin{aligned} \mathbf{z}^* = & \arg \min_{\mathbf{z}} J(\mathbf{z}) \\ \text{s.t. } & \mathcal{V}(\mathbf{z}, f) \subset \mathcal{P} \\ & \{s \geq 2 \frac{\max(r, h)}{d_f}\} \end{aligned} \quad (21)$$

- 5: **end if**

Algorithm 4 Function: $\mathcal{T} = \text{shortestPath}(G)$ Input: Graph G .Output: A sequence of valid configurations $\mathcal{T} = \{\mathbf{z}_s, \dots, \mathbf{z}_g\}$ and convex polytopes $\{\mathcal{P}_s, \dots, \mathcal{P}_g\}$ such that the robot team can navigate through them from the start to the goal, or \emptyset .

- 1: **if** \mathbf{z}_s and \mathbf{z}_g are **not** connected in G **then**
- 2: **return** \emptyset
- 3: **else**
- 4: **return** result of graph search on G where the cost of traversing an edge $E = \{\mathbf{z}_1, \mathbf{z}_2, \mathcal{P}\}$ is given by $d(\mathbf{z}_1, \mathbf{z}_2)$, the euclidean distance between the centroids of the two formations.
- 5: **end if**

intermediate setpoint for the team of robots. Denote its centroid by \mathbf{w} .

To reach the final goal, the team of robots sequentially follows the intermediate setpoints in the path and the local planner minimizes the deviation towards the associated configuration \mathbf{z}_w at every instance. To make progress towards the intermediate setpoints, and for improved performance, we slightly modify Algorithm 1 by selecting the convex region $\mathcal{P} \subset \bar{\mathcal{F}}^\tau(t_o)$ as follows.

We do not directly use the convex regions stored in the global path, since the robots need to account for dynamic obstacles in real-time.

First, a convex region $\mathcal{P}_{f_o \rightarrow w}$ containing all the robots in the team at their current positions, and directed towards \mathbf{w} is computed. If the setpoint \mathbf{w} is also inside the polytope, i.e. $\mathbf{z}_0, \mathbf{w} \in \mathcal{P}_{f_o \rightarrow w}$, then we use this polytope $\mathcal{P} := \mathcal{P}_{f_o \rightarrow w}$ for navigation.

Otherwise, we compute its intersection with a polytope generated with only the centroid of \mathbf{z}_0 and directed towards the waypoint \mathbf{w} , i.e. $\mathcal{P} := \mathcal{P}_{f_o \rightarrow w} \cap \mathcal{P}_{o \rightarrow w}$. In this case the robots may reconfigure to make progress towards the intermediate setpoint. See Section 2.4 for details on the computation of these polytopes.

5 Extension for mobile manipulators

In this section we describe an extension where a team of mobile manipulators collaboratively carry an object in a dynamic environment. To achieve this, the robot shape, formation definition and optimization equations are modified, and the derivations follow the same line of thought of the previous sections.

5.1 Robot and formation definition

The formation is defined by n mobile manipulators, each equipped with a robotic arm and grasping a rigid object at given points, see Fig. 5(b) for an example with two mobile manipulators.

The position $\mathbf{t} \in \mathbb{R}^2$ and orientation $\theta_o \in SO(2)$ of the object can vary. Each manipulator $i = \{1, \dots, n\}$ can rotate around the grasping point \mathbf{g}_i by an angle $\theta_i \in [\theta_{min}, \theta_{max}] \subset [-\pi/2, \pi/2] \subset SO(2)$ relative to the direction pointing towards the center of the object. Each manipulator may change the arm length, denote $a_i \in [a_{min}, a_{max}] \subset \mathbb{R}$ the distance from the center of the robot to the grasping position.

The vertices of the object relative to its center and expressed in the object coordinate frame are denoted by $\{\mathbf{w}_1^0, \dots, \mathbf{w}_{n_0}^0\}$. The vertices of manipulator i relative to its center and expressed in the robot coordinate frame are denoted by $\{\mathbf{w}_1^i, \dots, \mathbf{w}_{n_i}^i\}$. Denote the grasping positions on the object, relative to its center, by \mathbf{g}_0^i . Without loss of generality, in this derivation we assume that the angle between the robot base and arm is constant (additional degrees of freedom could be added to the set of optimization variables) and denote by \mathbf{a}_0^i the vector from the center of the robot to the grasping point, of length a_i , expressed in the robot coordinate frame. An example with three robots grasping a triangular object is shown in Figure 7.

Given the configuration $\mathbf{z} = [\mathbf{t}, \theta_o, a_1, \dots, a_n, \theta_1, \dots, \theta_n] \in \mathbb{R}^{3+2n}$ of the formation, the vertices of the manipulator and object, expressed in the world coordinate frame, are

$$\begin{aligned} \mathbf{v}_j^0 &= [\mathbf{t}, 0]^T + R_{\theta_o} \mathbf{w}_j^0, & \forall j \in [1, n_0] \\ \mathbf{v}_j^i &= [\mathbf{t}, 0]^T + R_{\theta_o} (\mathbf{g}_0^i + R_{\theta_i} (\mathbf{w}_j^i - \mathbf{a}_0^i)), & \forall j \in [1, n_i], \end{aligned} \quad (22)$$

where R_θ is the rotation matrix $[\cos(\theta), -\sin(\theta), 0; \sin(\theta), \cos(\theta), 0; 0, 0, 1]$. We denote by

$$\mathcal{V}(\mathbf{z}, i) = [\mathbf{v}_1^i, \dots, \mathbf{v}_{n_i}^i] \quad (23)$$

the set of n_i vertices for the object ($i = 0$) and each robot ($i > 0$) at configuration \mathbf{z} .

5.2 Obstacle-free convex region

Recalling Sec. 2.3, the position-time obstacle-free workspace is given by

$$\begin{aligned} \mathcal{O}^\tau(t_o) &= \mathcal{O} \times [0, \tau] \cup \bigcup_{\substack{t \in [0, \tau] \\ j \in \mathcal{I}_D}} \mathcal{D}_j(t_o + t) \times t \subset \mathbb{R}^4, \\ \mathcal{F}^\tau(t_o) &= \mathbb{R}^3 \times [0, \tau] \setminus \mathcal{O}^\tau(t_o) \subset \mathbb{R}^4. \end{aligned} \quad (24)$$

where, now, the static and dynamic obstacles are *not* dilated.

The collision-free convex polytope containing the robots at their current state and directed towards the goal is

$$\mathcal{P}_{f_o \rightarrow g} := \mathcal{P}_{\bigcup_{i=0:n} [\mathbf{v}_1^i(t_o), \dots, \mathbf{v}_{n_i}^i(t_o)] \times 0}^{[\mathbf{g}(t_1) \times \tau]}(\mathcal{F}^\tau(t_o)), \quad (25)$$

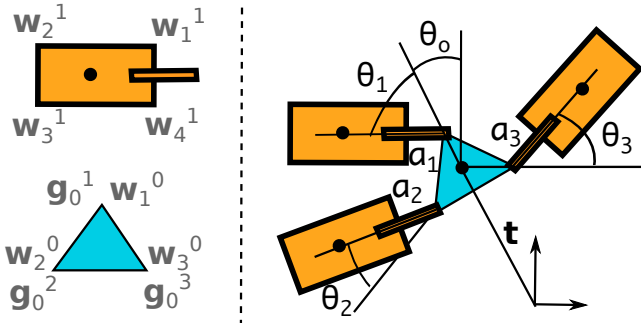


Figure 7. Right: optimization variables for three mobile manipulators grasping a triangular object. Left: vertices of a mobile manipulator and the grasped object. For this triangular object, the vertices are equal to the grasping points.

Additional polytopes in free space are computed analogously.

5.3 Non-linear optimization

Consider $\mathbf{z}_{ini} = [\mathbf{t}^{ini}, \theta_o^{ini}, a_1^{ini}, \dots, a_n^{ini}, \theta_1^{ini}, \dots, \theta_n^{ini}]$ the initial configuration of the robots and object at the current time.

Since the robots are rigidly attached to the object, we have to explicitly impose that the transition between the current and the target configuration remains within the convex polytope. Consider $K > 0$ interpolation steps, and denote by \mathbf{z}_λ the linearly interpolated configurations such that $\mathbf{z}_{\lambda=0} = \mathbf{z}_{ini}$ and $\mathbf{z}_{\lambda=K} = \mathbf{z}$. Angles are interpolated in the direction of minimum change and each interpolated configuration \mathbf{z}_λ is expressed as a function of \mathbf{z}_{ini} and \mathbf{z} , e.g. $\mathbf{t}_\lambda = \lambda(\mathbf{t} - \mathbf{t}^{ini})/K + \mathbf{t}^{ini}$.

Recalling Eq. (22) and the representation of the collision-free polytope \mathcal{P} by a set of linear constraints, as in Eq. (12), the optimization is

$$\begin{aligned} \mathbf{z}^* = & \arg \min_{\mathbf{z}} \|\mathbf{t} - \mathbf{g}(t_1)\|^2 \\ \text{s.t. } & \mathcal{V}(\mathbf{z}_\lambda, i) \times t_1 \subset \mathcal{P}_{f_o \rightarrow g} \\ & \theta_{min} \leq \theta_i \leq \theta_{max} \\ & a_{min} \leq a_i \leq a_{max} \\ & \forall j \in \{1, \dots, n_i\}, \forall i \in \{0, \dots, n\} \\ & \forall \lambda \in \{1, \dots, K\} \end{aligned} \quad (26)$$

The derivatives of the constraints with respect to the optimization variable \mathbf{z} are computed analogously to Eq. (17).

6 Results

In this section we present experiments with a team of three Kuka Youbot mobile manipulators collaboratively carrying an object and simulations with teams of quadrotor UAVs navigating in 3D environments. A video illustrating the results accompanies this paper.

The mobile manipulators are holonomic platforms. For the UAVs we employ the non-linear dynamical model and LQR controller used by [Alonso-Mora, Naegeli, Siegwart and Beardsley \(2015\)](#) with real quadrotors.

We use SNOPT by [Gill et al. \(2002\)](#) to solve the non-linear program via Sequential Quadratic Programming, a goal-directed version of IRIS by [Deits and Tedrake \(2014\)](#)

to compute the large convex regions and the Drake toolbox[‡] from MIT to handle quaternions.

6.1 Multiple aerial vehicles in formation

To evaluate our approach in 3D environments with aerial vehicles we present experiments in three simulated scenarios. The first scenario consists of four controlled quadrotors and four dynamic obstacles. The second scenario consists of four controlled quadrotors flying in formation and avoiding several static obstacles and one dynamic obstacle. The last scenario involves sixteen quadrotors flying in formation through a narrow corridor. In our visualizations we employ a cylinder since that is the shape we use for collision avoidance. Internally the quadrotors have an attitude controller and position controller and change their 3D pose within the enclosing cylinder, which is at all times kept vertical.

In all cases a new formation is computed every 2 s. The individual collision avoidance planners run at 5 Hz. The quadrotors move at speeds between 0.5 m/s and 1.5 m/s. In our simulations with four quadrotors a time horizon $\tau = 4$ s is considered. This is longer than the required time to reach a full stop. For the experiments with sixteen quadrotors a time horizon of $\tau = 10$ s is chosen, due to the large size of the formation and the scenario.

Consider the first scenario. Fig. 8 shows the trajectories of four quadrotors (in green and blue) passing through two lanes of dynamic obstacles (in yellow). The dynamic obstacles in the left lane move downwards at 0.4 m/s and the ones in the right move upwards with the same speed. Two default formations are considered, square (which is preferred) and diamond. The goal for the formation follows a constant velocity trajectory along the middle horizontal line and the team successfully adapts the parameters of the formation to remain collision-free and pass in-between the obstacles. In this case we imposed that the formation remains on the horizontal plane for illustrative purposes.

In order to evaluate the robustness of the method, we performed additional experiments for this first scenario for varying speed of the dynamic obstacles and the quadrotors flying in formation. The results are presented in Fig. 9. We observe that most of the time the target formation for the robots is within $\mathcal{P}_{f_o \rightarrow g}$, thus the formation is kept.

[‡]<http://drake.mit.edu>

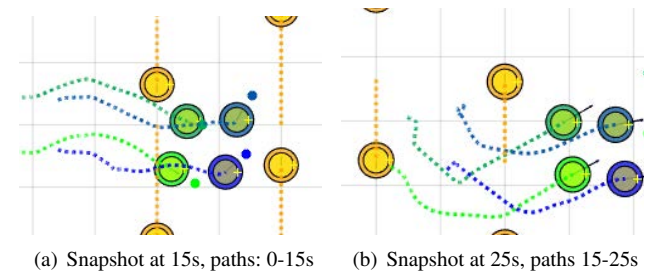


Figure 8. Top view. Four robots (green-blue) navigate a 8×15 m² environment with two lanes of dynamic obstacles (orange). The four robots locally reconfigure the formation moving towards the right side.

But, at high speeds, in order to quickly progress towards the goal, the robots temporally break it and select a target one within $\mathcal{P}_{o \rightarrow g}$ or \mathcal{P}_g . Good results, specially at lower dynamic obstacle speeds are observed. We believe that the results could be improved with an adaptive time horizon depending on the speed of both the moving obstacles and the formation. In scenarios with only static obstacle, the formation is maintained at all times.

In this scenario, very few collisions arise when the target speed of the formation is higher or similar to that of the dynamic obstacles and our framework successfully drives the robots towards the goal while avoiding collisions. Some collisions arise when the speed of the dynamic obstacles is much higher than that of the formation. This is due to the local planning horizon and the robots being unable to escape on time due to their lower speed. Again, this results may be improved with an adaptive time horizon of the framework.

Next, we present experiments for the second scenario. Fig. 10 shows snapshots and trajectories of four quadrotors tracking a circular trajectory while locally avoiding three static obstacles and a dynamic obstacle. Three default formations are considered here: square (1st preference), diamond (2nd preference) and line. The optimal parameters are computed with the non-linear optimization allowing rotation in 3D (flat horizontal orientation preferred) and reconfiguration.

The four quadrotors start from the horizontal square and slightly tilt it (11 s) to avoid the incoming dynamic obstacle.

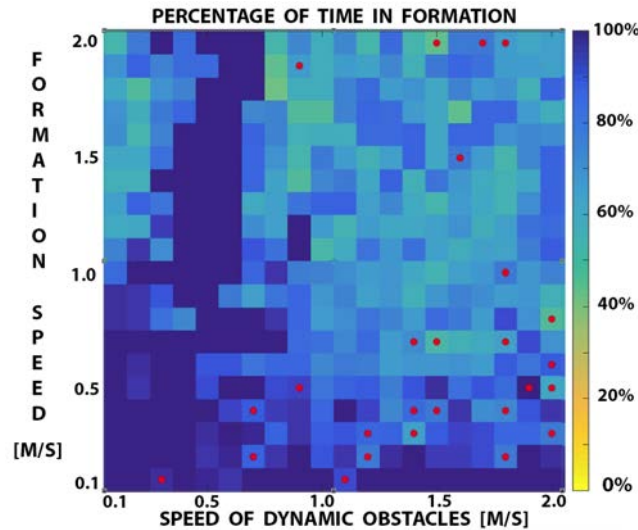


Figure 9. Results for 3D navigation in the scenario of Fig. 8 with four quadrotors and varying speed of the dynamic obstacles and the formation. Fixed time horizon of $\tau = 5$ s is used in all experiments. The figure shows the percentage of time in which the target formation is within $\mathcal{P}_{f_o \rightarrow g}$. This is, the percentage of time in which the formation is guaranteed to be maintained. If the optimization becomes unfeasible, due to the higher speed of the dynamic obstacles and the limited time horizon, to make progress towards the goal, the formation might be broken (This is, the target formation is found within $\mathcal{P}_{o \rightarrow g}$ or \mathcal{P}_g , but not in $\mathcal{P}_{f_o \rightarrow g}$). Scenarios for which a collision happened are marked with a red dot. As will be discussed in Section 7, collisions may arise mostly due to unforeseen changes of speed by moving obstacles or the limited time horizon.

To fully clear it while avoiding the obstacle in the lower corner, they shortly switch to a vertical line, and then back to the preferred square formation (20 s). To pass through the next narrow opening they switch back to the line formation (30 s) and then to the preferred square, tilted to avoid the dynamic obstacle (37 s). Once the obstacles are cleared they return to the preferred horizontal square formation (45 s).

Results of the third scenario, where sixteen quadrotors move along a corridor of three different widths are shown in Fig. 11. Three default formations are considered: 4x4x1 defined by four vertices (preferred), 4x2x2 defined by eight vertices and 8x2x1 defined by four vertices. At each time step the method computes the optimal parameters for each of the three and selects the one of lowest cost. Between times 75 s and 110 s the method successfully rotates the formation by 90° for it to be collision free (the default formations were horizontal, which is also preferred in the cost function).

Thanks to the abstraction of a formation by the vertices of its convex hull, see Sec. 2.5, the computation time of the non-linear optimization is independent of the number of robots - as long as the same convex shape is maintained - and can be executed in real-time. It is worth noting that in this algorithm the dimension of the space where the robots move has little influence in the computational cost, which depends mostly on the number of variables defining the formation. In Table 2 we provide computational times for our implementation using a 2.6 GHz i7 laptop. The approach shows close to real time performance, typically below 300 milliseconds.

6.2 Collaborative transport with two mobile manipulators

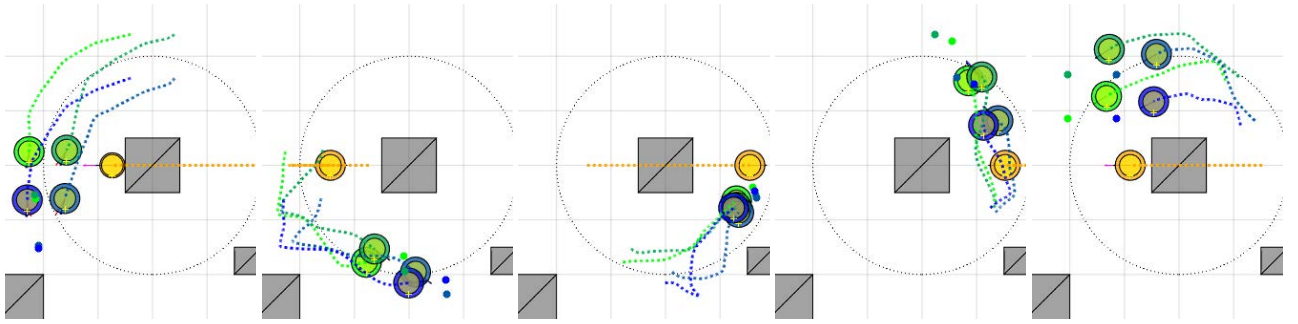
We performed initial experiments with two mobile manipulators carrying a rigid object, as described in Sec. 5. In this first experiment the two robots are not allowed to change their orientation and distance with respect to the object ($\theta_{i=\{1,2\}} = 0$, $\mathbf{a}_{i=\{1,2\}} = \text{constant}$). We optimize for the position and orientation of the object only.

Four snapshots are shown in Fig. 12 of an experiment where the two mobile manipulators successfully carry the rigid object to the goal position behind the orange boxes while locally avoiding collisions with the human. In all of our experiments, executed with external tracking, the robots successfully adapted their formation to avoid collisions. This assumes that the human cooperates, otherwise, collisions may still occur if the human moves faster than the robots or traps them against an obstacle.

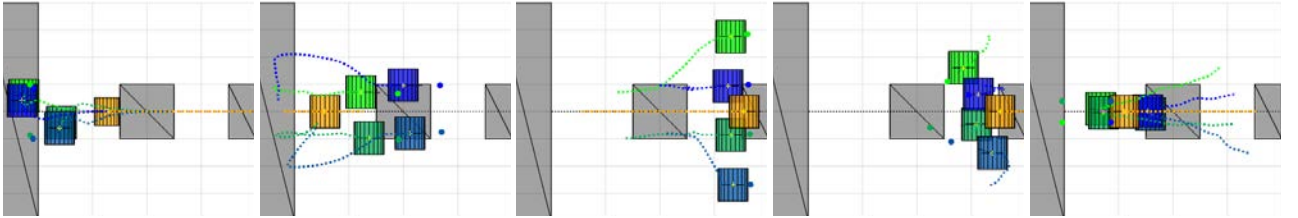
A visualization with convex regions of another experiment is shown In Fig. 13. For each snapshot the current (blue) and target (green) formation given by the optimization are displayed. The two robots successfully adapt their formation, rotating as required, to avoid both the dynamic obstacle (red) and static (grey) obstacles in this 8m x 6m scenario. Slices

Table 2. Computational time [ms] for our implementation

Compute	Min	Mean	Max	Std deviation
Convex region	31.8	82.8	221.4	72.1
NL Optimization	93.7	226.4	522.7	64.1

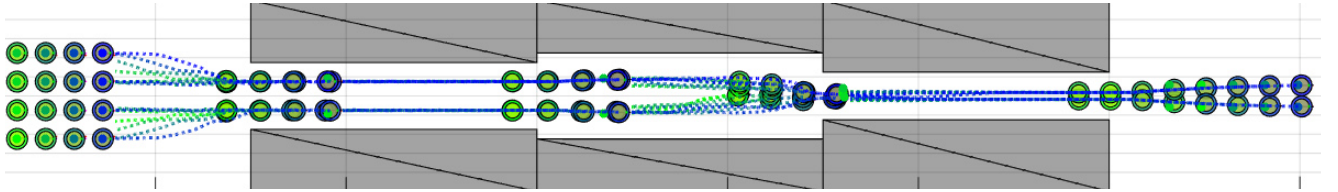


(a) Top view. From left to right, snapshots at 11 s, 20 s, 30 s, 37 s and 45 s, and paths of the robots in-between.

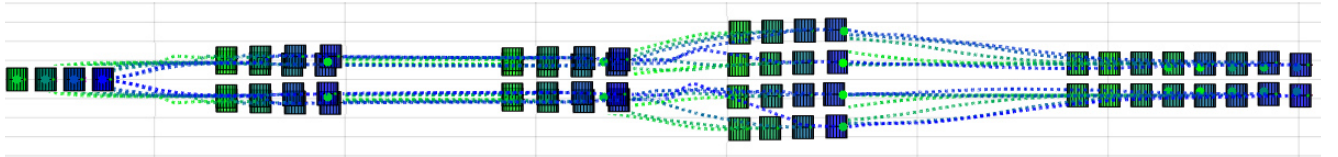


(b) Side view for the same snapshots of above. All five are obtained from a camera placed at the lower side of the top view. For clarity, the quadrotors are always displayed on top of obstacles even if they might be behind.

Figure 10. Four quadrotors (green-blue cylinders) navigate in a $12 \times 12 \times 6 \text{ m}^3$ scenario with three static obstacles (grey) and a dynamic obstacle (yellow). The four quadrotors track a circular motion (black dots in top view) and locally reconfigure the formation to avoid collisions and make progress.



(a) Top view with robot paths and snapshots at time 0 s, 25 s, 50 s, 75 s, 110 s. Robots move from left to right.



(b) Side view with robot paths and snapshots at time 0 s, 25 s, 50 s, 75 s, 110 s. Robots move from left to right.

Figure 11. 16 quadrotors navigate along a $70 \times 10 \times 10 \text{ m}$ corridor, with obstacles shown in grey. The quadrotors locally adapt the formation to remain collision free. The following formations are observed: $4 \times 4 \times 1$ - $4 \times 2 \times 2$ - $4 \times 4 \times 1$ (vertical) - $8 \times 2 \times 1$ (vertical), finally transitioning towards horizontal $8 \times 2 \times 1$.



Figure 12. Four consecutive snapshots of an avoidance maneuver where two mobile manipulators collaboratively carry a rigid object and navigate it to the goal while adjusting their formation to avoid collisions with the orange boxes and the human. Robots and human are tracked by overhead cameras. This maneuver is performed in 1 minute.

at t_o and $f_f = t_o + \tau$ of the convex region computed in position-time space are also shown for illustrative purposes. A time horizon $\tau = 2 \text{ s}$ was employed.

6.3 Collaborative transport, three mobile manipulators

We performed additional experiments with three mobile manipulators carrying an object, as described in Sec. 5. All three robots are allowed to change their orientation

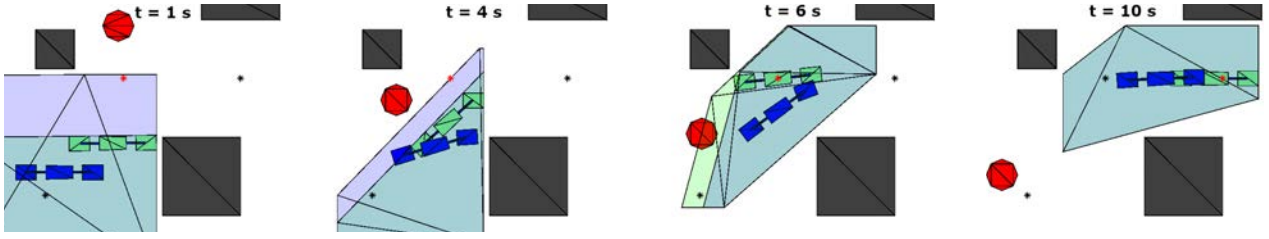


Figure 13. Four consecutive snapshots of a 10 s avoidance maneuver where two mobile manipulators collaboratively carry a rigid object and navigate to two goals (crosses) while avoiding collisions with static (grey) and dynamic (red hexagon) obstacles. The current state of the two manipulators and the rigid object is displayed in blue and the target one (given by the optimization) in green. Two slices of the convex polytope are shown, purple for the current time $t_o = t$ (shown in the figure titles) and light green for time $t_1 = t_o + \tau$ (the intersection is the larger blueish region). The dynamic obstacle is shown at time t_o and it is moving at constant speed downwards. As displayed, the red dynamic obstacle may intersect with the light green slice of the convex polytope (at t_1), but not with the purple one (at t_o). The manipulators successfully navigate the rigid object through the two set points avoiding collisions. The initial, intermediate and final setpoints are shown with dots, the currently active one in red, the others in black.

respect to the object, but their distance remains constant. We optimize for the position and orientation of the object and the orientation of all three manipulators. Given a goal for the formation, we first compute a global path with the algorithm of Sec. 4 considering only the two static obstacles. The local motion planner then runs at a frequency of approximately 5 Hz, accounts for the dynamic obstacle (person), and updates the parameters of the formation. A low level controller is employed which, via high-frequency interpolation, drives the robots towards the desired formation.

We tested different configurations of the two boxes in our experimental space, covering all possible scenarios we thought of (some examples are shown in Fig. 14). In all of our experiments the robots were able to avoid collisions and reach the goal - as long as the human moved at a reasonable speed below that of the robots and did not aggressively push them against a wall.

Several configurations of the two boxes, with the computed global path, are shown in Fig. 14. All of them were computed in the order of below ten seconds. The initial configuration is in the lower part of the images and the goal configuration in the upper part. In each figure we display the samples (black dots), the convex regions (blue), the optimal formations within each intersection (green) and the path. We stop the construction of the graph as soon as the first solution is found. We observe that in general, very few iterations were required to find a feasible solution, which is also of good quality thanks to the optimizer. Navigation in all these scenarios was successfully achieved by the three mobile manipulators.

A representative experiment where the three robots navigate through the boxes and avoid a human is shown In Fig. 15. For reference, in Fig. 16 we show twelve different scenarios and the configuration of the three robots when navigating through the environment.

In these experiments we employed a triangular object with foam exterior. The foam provides a small degree of deformability to compensate for the lack of compliance in the robot arms and low level controller of the mobile manipulators. Note that successful manipulation of a perfectly rigid body was shown in the previous experiments with two mobile manipulators, albeit at lower speeds.

6.4 Global planning in large scenarios

We also tested the approach in several larger scenarios. In Fig. 17 we show two examples of the global planner in simulated 2D environments. In these two cases the global planner is allowed to run for several iterations, up to a fixed amount of time. We display all the computed convex regions and formations. After finding the first feasible path connecting the start with the goal position, we store (and display) the subsequent shorter paths found by the algorithm. An advantage of the method is that large areas in free space are explored by each convex polytope, which reduces the need for additional samples within.

7 Discussion

The method described in this paper showed good performance, with real-time performance and was able to successfully compute the optimal parameters for the multi-robot formation, while allowing for reconfiguration. The method provided collision-free navigation among static and dynamic obstacles in simulations with aerial vehicles and in experiments with mobile manipulators.

At least in part, the computational efficiency and the good scalability with respect to the number of robots in the formation is achieved by (a) not including the agent dynamics in the formation optimization but handling them in the individual local planners (this works well for robots with fast dynamics); and (b) considering the convex hull of the formation. In fact, the number of variables and constraints of the original formation control method is independent from the number of robots. The optimization problem in Eq. (15) has eight variables for 3D motion and $n_i \cdot n_l + 2$ constraints, where n_i is the number of vertices of the convex hull of robot positions in formation i and n_l is the number of sides in the convex polytope. We recall that the number of vertices of the convex hull depends on the shape of the formation, e.g. a square formation has four vertices, independently of the number of robots therein. For collaborative manipulators, the number of variables ($3 + 2n$), and constraints ($2n + m \cdot n_l \cdot (n_o + n_i \cdot n)$), scale linearly with the number of robots. In this case, n_o and n_i are the number of vertices of the object and each robot, respectively.

In our experiments, the computation of a large obstacle-free convex polytope following Sec. 2.4 showed very good

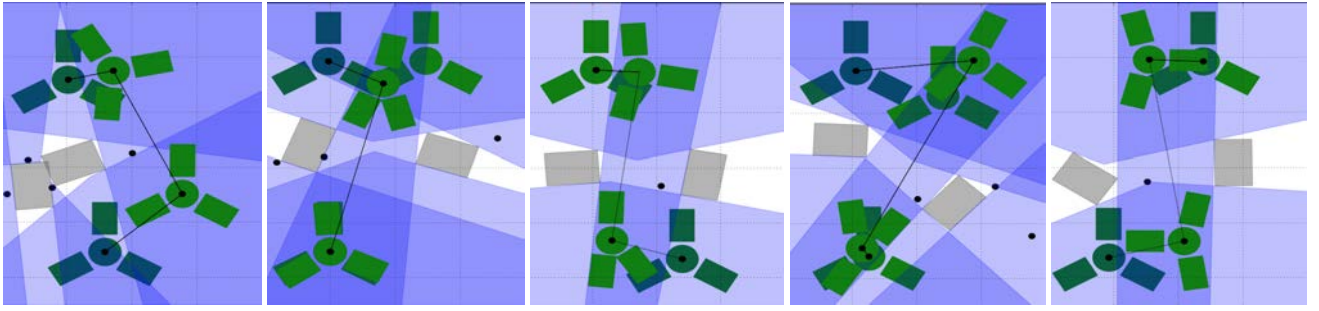
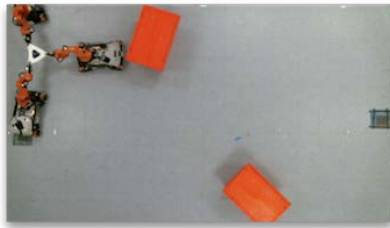


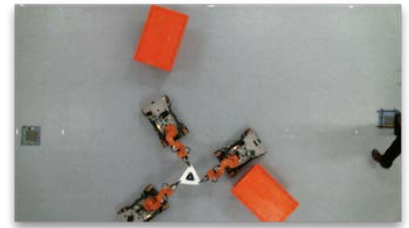
Figure 14. Five examples of the global path planner for different scenarios with two static obstacles. The three mobile manipulators carry an object and are able to rotate and translate while grasping. The initial and final formations are displayed in dark green. Light green formations are additional nodes of the graph. The first feasible path is displayed with a solid black line. All samples (black dots), polytopes (blue) and optimized formations (green) within the intersections of polytopes are shown. Typically a feasible path is found with very few samples.



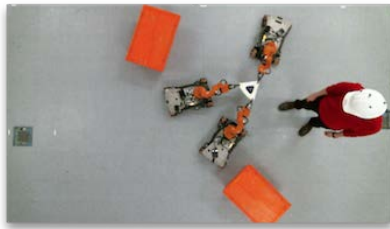
(a) Time 0s



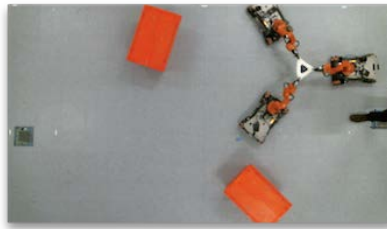
(b) Time 13s



(c) Time 35s



(d) Time 48s



(e) Time 60s



(f) Time 80s

Figure 15. Three mobile manipulators collaboratively carry an object and navigate to a goal position in the other side of the room. The global path planner guides them through the two static obstacles and they locally avoid the walking human. The robots successfully adapt their formation to pass through the narrow opening and avoid the human.

results, but no guarantees exist that the best volume will be obtained. In fact the method will converge to a local optimum of the cost function, which is guaranteed to be fully contained in free space. Searching over several regions might prove advantageous. One may also consider employing a faster, albeit suboptimal algorithm to quickly compute a convex region.

To compute the parameters of the multi-robot formation our method solves a non-linear optimization via Sequential Convex Programming. This method converges to a local optimum of the non-convex problem. Global optimality can only be guaranteed if the original optimization problem is convex, which is typically not the case. For the non-convex case, the number of iterations required to find a locally optimal, or even feasible, solution is not defined. In practice, the method performed very well, quickly returning good parameters for the formation in all cases where a valid formation could be fitted within the convex polytope.

These observations also apply to the case of the global planner and no strong guarantees can be given for the general non-convex optimization case. Thanks to the sampling of

convex regions, the method will successfully explore the whole workspace. For speed-up, as described in Algorithm 2, we limit the sampling of regions to points outside of the union of current convex regions in the graph. In most scenarios this heuristic works well, but it can potentially miss narrow openings, since, although the whole space is covered by convex regions, the intersection might not be traversable. Two advantages of this method are (a) that sampling is performed in a low dimension space - the workspace - instead of in the high-dimensional configuration space. And, (b) that large areas of the free space are explored/covered at once when contained within a convex polytope. This has the potential to speed-up global path planning for formations of robots.

If the optimizations are feasible and a solution is found, the motion is guaranteed to be collision-free up to the time horizon of the local planner, under the assumption that the moving obstacles maintain their speeds constant. This is true because: (a) the convex region is fully contained in free position-time space, (b) the robots at their initial position and at the positions in the target formation are fully contained

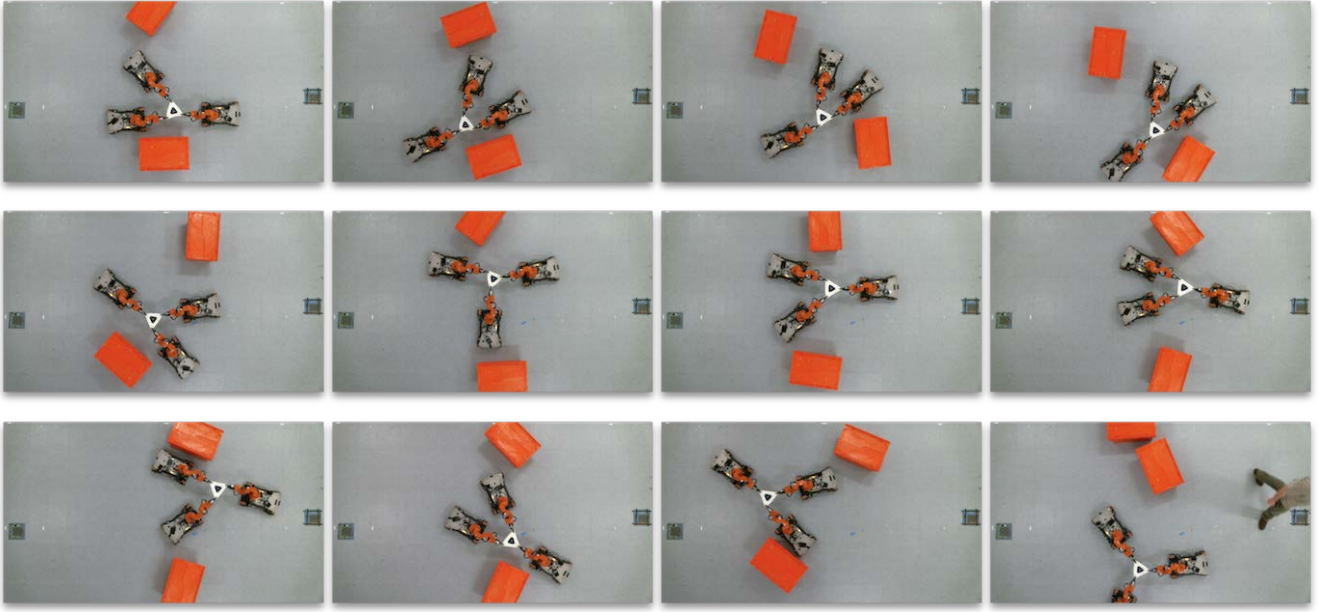
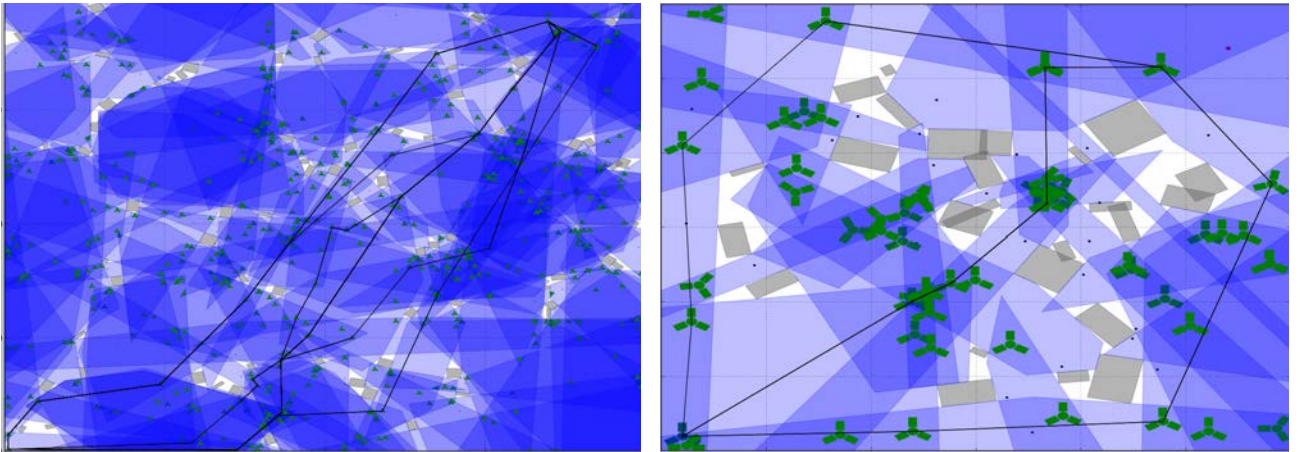


Figure 16. Twelve different experiments where the three mobile manipulators collaboratively carry an object and navigate to a goal position in the other side of the room. For each one of them, a snapshot while they traverse through the two obstacles is shown. This experiment shows robustness to the location of the obstacles and that the robot formations vary across different execution runs. If the obstacles leave enough free space, as is the case in the lower right corner experiment, the team of robots maintain the preferred formation and do not need to rotate around their grasping points. Otherwise, they successfully adapt the configuration.



(a) Large 100 m x 100 m scenario with rectangular obstacles of size between 1 m² and 25 m². The algorithm run for 100 seconds.

(b) Medium 15 m x 15 m scenario with rectangular obstacles of size between 0.25 m² and 9 m². The algorithm run for 15 seconds.

Figure 17. Two examples of the global path planner connecting a start (lower left corner) with a goal (upper right corner) for two large scenarios with many static obstacles. The three mobile manipulators carry an object and are able to rotate and translate while grasping. The algorithm runs for a fixed amount of time. The first feasible path, and the feasible paths found in subsequent iterations that decrease the cost, are displayed with a solid black line. All samples (black dots), polytopes (blue) and optimized formations (green) within the intersections of polytopes are shown.

in the convex region, and (c) the motion in between the two formations as well, if the robots move in a straight line (the linear combination of two points lies within the convex polytope). For mobile manipulators collaboratively carrying an object, this is satisfied up to the interpolation.

Nonetheless, collisions with moving obstacles can still arise if the assumptions are not met. For instance, if the moving obstacles change the direction of motion quicker than the robots can react, if the moving obstacles move too fast, if the planning horizon is not long enough or if the team

of robots are trapped in a corner from where they can not feasibly escape.

An advantage of the method is that planning is decoupled into: (a) finding convex regions in the lower dimensional free position-time space (\mathbb{R}^4) and (b) efficiently optimizing the configuration of the team of robots within those convex regions. This comes at the expense of completeness, since in our approach we require that the robot team maintains a formation that does not intersect with obstacles, i.e. the robots can not maintain a formation while letting an obstacle pass through. In the event of dynamic obstacles, the team

may break the formation to let a moving obstacle pass through, and come back to the original formation as soon as there is enough free room.

Lastly, the method is general and can be adapted to other high-dimensional problems or formation definitions. The core idea of the algorithms is to generate convex obstacle-free regions and then optimize the parameters of the formation (i.e. the degrees of freedom of the high-dimensional configuration) such that the robots are fully contained in the convex region. The only requirements to adapt the method are (a) a function that converts configurations \mathbf{z} to the outer vertices of the formation $\mathcal{V}(\mathbf{z}, f)$ or high-dimensional system, and (b) a way to compute derivatives with respect to the configuration \mathbf{z} (unless they are computed numerically).

8 Conclusion

In this paper we showed that navigation of teams of robots in formation among arbitrary static and dynamic obstacles can be achieved via a constrained non-linear optimization. By first computing a large obstacle-free convex polytope and then optimizing the formation parameters, low computational cost is achieved together with good navigation results. In several simulations with aerial vehicles navigating in 3D environments we showed successful navigation in formation where robots may reconfigure the formation as required to avoid collisions and make progress.

Our method can be applied both for real-time local navigation in a dynamic environment and to compute global paths in static environments. The global planner successfully combines a sampling-based method in the workspace with non-linear optimization for the remaining degrees of freedom of the formation, thus reducing the dimensionality of the sampling problem.

For formation control, the approach scales to teams of robots of arbitrary size, since only the convex hull of the formation is employed in the constrained optimization. Simulations with sixteen quadrotors -although more could be used - demonstrate this. The approach is general and can also be adapted to other formation definitions and applications, as showed in our experiments with three mobile manipulators collaboratively carrying an object,

In this work we did allow for splitting and merging of robots, from/to a joint formation to/from individual navigation. An interesting avenue for future work is that of splitting and merging of the group formation into smaller sub-formations, or to maintain the formation while letting dynamic obstacles through, which is currently not possible. Additional avenues of future research include incorporating the dynamic constraints of the robots in the non-linear optimization problem and accounting for uncertainties in the prediction of the movement of the dynamic obstacles. In this work the non-linear dynamics of the robots were decoupled from the formation control and accounted for by the individual controllers locally.

Acknowledgements

The authors are grateful to Robin Deits and Hongkai Dai from MIT for their help with IRIS, Drake and SNOPT.

Funding

This work was supported in part by the MIT Lincoln Laboratory, SMARTS N00014-09-1051, pDOT ONR N00014-12-1-1000 and the Boeing Company. We are grateful for their support.

Appendix: Collision Avoidance

In this appendix we provide a description of the method for collision avoidance employed with the aerial vehicles. We implement the convex optimization introduced by [Alonso-Mora, Naegeli, Siegwart and Beardsley \(2015\)](#) with identical motion constraints and constraints for avoidance of other agents. This approach adapts to changes in the environment, avoids moving obstacles and respects the dynamics of the robot via a set of motion primitives. Each motion primitive was defined to track a constant reference velocity with a robot-specific controller.

We extend the method towards environments with complex static obstacles. In particular, using the convex polytope computation described in Sec. 2.4, we add a new constraint to guarantee that the motion of the robot is within the obstacle-free workspace \mathcal{F} . Following the notation of [Alonso-Mora, Naegeli, Siegwart and Beardsley \(2015\)](#), the additional constraint for avoidance of static obstacles is computed as follows.

Denote by $\bar{\mathcal{O}}^\epsilon = \{\mathbf{p} \in \mathbb{R}^3 \mid \mathcal{A}_\epsilon(\mathbf{p}) \cap \mathcal{O} \neq \emptyset\}$ the set of static obstacles dilated by the robot volume plus a small value $\epsilon > 0$. A convex polytope $\mathcal{P}_{\mathbf{p}_i}^{\mathbf{r}_{\sigma(i)}}(\mathbb{R}^3 \setminus \bar{\mathcal{O}}^\epsilon) \subset \mathbb{R}^3$ is computed following Sec. 2.4. This polytope is in obstacle-free space, contains the initial position \mathbf{p}_i of the robot, and is directed towards the robot's goal position $\mathbf{r}_{\sigma(i)}$ in the new formation.

In [Alonso-Mora, Naegeli, Siegwart and Beardsley \(2015\)](#) the collision avoidance algorithm was formulated as a constrained optimization in velocity space. Therefore, the convex region needs to be converted to an equivalent region in velocity space. Given the time horizon τ of the planner, this is formally

$$\mathcal{P}^u(\mathbf{p}, \epsilon) := (\mathcal{P}_{\mathbf{p}_i}^{\mathbf{r}_{\sigma(i)}}(\mathbb{R}^3 \setminus \bar{\mathcal{O}}^\epsilon) - \mathbf{p}_i) / \tau, \quad (27)$$

where each linear constraint defining $\mathcal{P}_{\mathbf{p}_i}^{\mathbf{r}_{\sigma(i)}}(\mathbb{R}^3 \setminus \bar{\mathcal{O}}^\epsilon)$ is expressed relative to the current position of the robot and is divided by the time horizon. In particular, if the robot selects a reference velocity that satisfies this constraint, i.e. $\mathbf{u} \in \mathcal{P}^u(\mathbf{p}, \epsilon)$, then all future positions up to the time horizon τ are within $\mathcal{P}_{\mathbf{p}_i}^{\mathbf{r}_{\sigma(i)}}(\mathbb{R}^3 \setminus \bar{\mathcal{O}}^\epsilon)$. This polytope is then included in the distributed convex optimization of [Alonso-Mora, Naegeli, Siegwart and Beardsley \(2015\)](#).

If the target position $\mathbf{r}_{\sigma(i)}$ of robot i is within its line of sight (this is the case if $\mathbf{z} \in \mathcal{P}_{f_o \rightarrow g}$), then the collision avoidance algorithm successfully drives the robot towards it. Otherwise, a global planner, such as the ones proposed by [Bento et al. \(2013\)](#) or [Yu and LaValle \(2013\)](#), can be used for guidance.

References

- Alonso-Mora, J., Baker, S. and Rus, D. (2015). Multi-robot navigation in formation via sequential convex programming, *IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*.

- Alonso-Mora, J., Knepper, R. A., Siegwart, R. and Rus, D. (2015). Local Motion Planning for Collaborative Multi-Robot Manipulation of Deformable Objects, *IEEE International Conference Robotics and Automation*.
- Alonso-Mora, J., Montijano, E., Schwager, M. and Rus, D. (2016). Distributed Multi-Robot Navigation in Formation among Obstacles: A Geometric and Optimization Approach with Consensus, *IEEE International Conference on Robotics and Automation (ICRA)*.
- Alonso-Mora, J., Naegeli, T., Siegwart, R. and Beardsley, P. (2015). Collision avoidance for aerial vehicles in multi-agent scenarios, *Autonomous Robots* **39**(1): 101–121.
- Alonso-Mora, J., Schoch, M., Breitenmoser, A., Siegwart, R. and Beardsley, P. (2012). Object and animation display with multiple aerial vehicles, *IEEE/RSJ International Conference on Intelligent Robots and Systems*.
- Augugliaro, F., Schoellig, A. P. and D’Andrea, R. (2012). Generation of collision-free trajectories for a quadcopter fleet: A sequential convex programming approach, *IEEE/RSJ International Conference on Intelligent Robots and Systems*.
- Ayanian, N., Kallem, V. and Kumar, V. (2011). Synthesis of feedback controllers for multiple aerial robots with geometric constraints., *IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*.
- Ayanian, N., Kumar, V. and Koditschek, D. E. (2009). Synthesis of Controllers to Create, Maintain, and Reconfigure Robot Formations with Communication Constraints., *International Symposium on Robotics Research (ISRR)*.
- Balch, T. and Arkin, R. C. (1998). Behavior-based formation control for multirobot teams, *IEEE Trans. on Robotics and Automation* **14**(6): 926–939.
- Balch, T. and Hybinette, M. (2000a). Social potentials for scalable multi-robot formations, *IEEE International Conference Robotics and Automation*.
- Balch, T. and Hybinette, M. (2000b). Social potentials for scalable multi-robot formations, *IEEE International Conference on Robotics and Automation (ICRA)*.
- Barfoot, T. D. and Clark, C. M. (2004). Motion planning for formations of mobile robots., *Robotics and Autonomous Systems* **46**: 65–78.
- Belta, C. and Kumar, V. (2004). Abstraction and Control for Groups of Robots, *IEEE Transactions on Robotics* **20**(5): 865–875.
- Bento, J., Derbinsky, N., Alonso-Mora, J. and Yedidia, J. S. (2013). A message-passing algorithm for multi-agent trajectory planning, *Advances in neural information processing systems NIPS*.
- Bertsekas, D. P. (1988). The auction algorithm: a distributed relaxation method for the assignment problem, *Ann. Oper. Res.* **14**(1): 105–123.
- Cheah, C. C., Hou, S. P. and Slotine, J. (2009). Region-based shape control for a swarm of robots, *Automatica* **45**: 2406–2411.
- Chen, Y., Cutler, M. and How, J. P. (2015). Decoupled multiagent path planning via incremental sequential convex programming, *IEEE International Conference on Robotics and Automation (ICRA)*.
- Chen, Y. Q. and Wang, Z. (2005). Formation control: a review and a new consideration, *IEEE/RSJ International Conference on Intelligent Robots and Systems*.
- Conner, D. C., Rizzi, A. A. and Choset, H. (2003). Composition of local potential functions for global robot control and navigation., *IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*.
- Cortés, J. (2009). Global and robust formation-shape stabilization of relative sensing networks., *Automatica* **45**(12): 2754–2762.
- Deits, R. and Tedrake, R. (2014). Computing large convex regions of obstacle-free space through semidefinite programming, *Workshop on the Algorithmic Fundamentals of Robotics*.
- Deits, R. and Tedrake, R. (2015). Efficient mixed-integer planning for UAVs in cluttered environments, *IEEE International Conference on Robotics and Automation (ICRA)*.
- Derenick, J. C. and Spletzer, J. R. (2007). Convex Optimization Strategies for Coordinating Large-Scale Robot Formations., *IEEE Transactions on Robotics* **23**: 1252–1259.
- Derenick, J., Spletzer, J. and Kumar, V. (2010). A semidefinite programming framework for controlling multi-robot systems in dynamic environments, *IEEE Conference on Decision and Control*.
- Desai, J. P., Ostrowski, J. P. and Kumar, V. (2001). Modeling and control of formations of nonholonomic mobile robots, *IEEE Trans. on Robotics and Automation* **17**(6): 905–908.
- Dimarogonas, D. V. and Johansson, K. H. (2008). On the stability of distance-based formation control, *IEEE Conference on Decision and Control (CDC)*.
- Dimarogonas, D. V. and Kyriakopoulos, K. J. (2005). Formation Control and Collision Avoidance for Multi-Agent Systems and a Connection between Formation Infeasibility and Flocking Behavior, *IEEE Conference on Decision and Control (CDC)*.
- Dunbar, W. B. and Murray, R. M. (2002). Model predictive control of coordinated multi-vehicle formations, *IEEE Conference on Decision and Control*.
- Egerstedt, M. and Hu, X. (2001). Formation constrained multi-agent control, *IEEE Transactions on Robotics and Automation* **17**(6): 947–951.
- Erdmann, M. and Lozano-Perez, T. (1987). On multiple moving objects, *Algorithmica* **2**: 477–521.
- Fax, J. A. and Murray, R. M. (2004). Information flow and cooperative control of vehicle formations, *Automatic Control, IEEE Transactions on* **49**(9): 1465–1476.
- Fredslund, J. and Mataric, M. J. (2002). A general algorithm for robot formations using local sensing and minimal communication, *IEEE Transactions on Robotics and Automation* **18**(5): 837–846.
- Gill, P. E., Murray, W. and Saunders, M. A. (2002). SNOPT: An SQP algorithm for large-scale constrained optimization, *SIAM journal on optimization* **12**(4): 979–1006.
- Hsieh, A., Kumar, V. and Chaimowicz, L. (2008). Decentralized controllers for shape generation with robotic swarms, *Robotica* **26**: 691–701.
- Kallem, V., Komoroski, A. T. and Kumar, V. (2011). Sequential Composition for Navigating a Nonholonomic Cart in the Presence of Obstacles, *IEEE Transactions on Robotics* **27**(6): 1152–1159.
- Khatib, O., Yokoi, K., Chang, K., Ruspini, D., Holmberg, R. and Casal, A. (1996). Vehicle/arm coordination and multiple mobile manipulator decentralized cooperation, *Proceedings of the 1996 IEEE/RSJ International Conference on Intelligent Robots and Systems*.

- Kloder, S. and Hutchinson, S. (2006). Path planning for permutation-invariant multirobot formations, *Robotics, IEEE Transactions on* **22**(4): 650–665.
- Krontiris, A., Louis, S. and Bekris, K. E. (2012). Multi-level formation roadmaps for collision-free dynamic shape changes with non-holonomic teams, *IEEE International Conference on Robotics and Automation (ICRA)*.
- Kuhn, H. W. (1955). The hungarian method for the assignment problem, *Naval Research Logistics*, pp. 83–97.
- Kushleyev, A., Mellinger, D. and Kumar, V. (2012). Towards a swarm of agile micro quadrotors, *Robotics: Science and Systems*.
- Latombe, J. C. (1991). *Robot Motion Planning*, Boston: Kluwer.
- LaValle, S. M. (2006). *Planning Algorithms*, Cambridge University Press.
- LaValle, S. M. and Kuffner, J. J. (2001). Randomized kinodynamic planning, *The International Journal of Robotics Research* **20**(5): 378–400.
- Michael, N. and Kumar, V. (2008). Controlling Shapes of Ensembles of Robots of Finite Size with Nonholonomic Constraints, *Proceedings of Robotics: Science and Systems IV*.
- Michael, N., Zavlanos, M. M., Kumar, V. and Pappas, G. J. (2008). Distributed multi-robot task assignment and formation control, *of the IEEE International Conference on Robotics and Automation*.
- Morgan, D., Subramanian, G. P., Chung, S. J. and Hadaegh, F. Y. (2016). Swarm assignment and trajectory optimization using variable-swarm, distributed auction assignment and sequential convex programming, *The International Journal of Robotics Research* **35**(10): 1261–1285.
- Ogren, P., Egerstedt, M. and Hu, X. (2001). A control Lyapunov function approach to multi-agent coordination, *IEEE Conference on Decision and Control*.
- Olfati-Saber, R. and Murray, R. M. (2002). Distributed cooperative control of multiple vehicle formations using structural potential functions, *IFAC World Congress*.
- Ren, W. and Sorensen, N. (2008). Distributed coordination architecture for multi-robot formation control, *Robotics and Autonomous Systems* **56**(4): 324–333.
- Sabattini, L., Secchi, C. and Fantuzzi, C. (2011). Arbitrarily shaped formations of mobile robots: artificial potential fields and coordinate transformation, *Autonomous Robots* **30**: 385–397.
- Saha, I., Ramaithitima, R., Kumar, V., Pappas, G. J. and Seshia, S. A. (2014). Automated Composition of Motion Primitives for Multi-Robot Systems from Safe LTL Specifications, *IEEE/RSJ International Conference on Intelligent Robots and Systems*.
- Sugar, T. G. and Kumar, V. (2002). Control of cooperating mobile manipulators, *Robotics and Automation, IEEE Transactions on* **18**(1): 94–103.
- Tang, C. P., Bhatt, R. and Krovi, V. N. (2004). Decentralized kinematic control of payload transport by a system of mobile manipulators, *IEEE International Conference on Robotics and Automation*.
- Tanner, H. G., Jadbabaie, A. and Pappas, G. J. (2007). Flocking in Fixed and Switching Networks, *IEEE Transactions on Automatic Control* **52**(5): 863–868.
- Tanner, H. G., Loizou, S. G. and Kyriakopoulos, K. J. (2003). Nonholonomic navigation and control of cooperating mobile manipulators, *Robotics and Automation, IEEE Transactions on* **19**(1): 53–64.
- Yu, J. and LaValle, S. M. (2013). Fast, near-optimal computation for multi-robot path planning on graphs, *AAAI Conference on Artificial Intelligence, late breaking papers*.
- Zhou, D. and Schwager, M. (2015). Virtual Rigid Bodies for coordinated agile maneuvering of teams of micro aerial vehicles, *IEEE International Conference on Robotics and Automation (ICRA)*.